THE TEACHING OF MATHEMATICS IN COLLEGES AND UNIVERSITIES*

E. P. VANCE, Oberlin College

1. Introduction. In the early part of 1947 the President of the United States requested a report on science teaching in the colleges and universities of this country; he desired information on the kinds of research being done both in and out of government, where research was being done, how it was being financed and administered, and so on. In connection with this request the President's Scientific Research Board asked the American Association for the Advancement of Science to study the effectiveness of science teaching in colleges and universities. This request was turned over to the Cooperative Committee on Science Teaching sponsored by the A.A.A.S.† The findings of this Committee have formed an important part of the report made by the Scientific Research Board on our scientific personnel resources.‡ The Board also gathered data on total enrollment in science classes, the number of scientists now employed in various types of institutions, and other matters related to the development of scientific talent.

In order to obtain necessary information the Cooperative Committee decided to solicit data and opinions from chairmen of science departments in colleges and universities. Thus the Scientific Research Board sent out a request for information in the form of a comprehensive questionnaire to two hundred and fifty such chairmen, five at each of fifty colleges and universities. Since these fifty institutions had granted more than ninety percent of the Ph.D.'s in science during the past ten years, the composite replies constitute a significant contribution to the study of the effectiveness of science teaching and a general view of the over-all picture as it occurs today. Forty-one of the fifty institutions which received requests for information in mathematics replied, some in more detail than others. The purpose of this article is to summarize those findings which seem to be of interest to mathematicians.

The questions on the questionnaire were general in character and were placed in seven different categories, each of which was believed to represent an area of problems pertinent to the effectiveness of science teaching. Since the same questionnaire was used for all the sciences, some of the topics mentioned were, of course, not particularly apposite to the field of mathematics.

2. The size of enrollments. In the undergraduate schools during the year 1946–47, it was generally true that there were between two and three times the number of students enrolled in mathematics in 1940. Only one school re-

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† Professor Raleigh Schorling of the University of Michigan is the representative of the M.A.A. on this Committee.

‡ See Manpower for Research, Vol. IV of Science and Public Policy, Oct. 11, 1947, U. S. Government Printing Office, Washington 25, D. C., Price 35 cents.

ported no increase, while the largest increase was five hundred percent at one of the large western universities. In many cases the enormous influx of students made it necessary to appoint as regular instructors in mathematics a large number of individuals who would not have been appointed under ordinary conditions. Such teaching personnel would be unable to teach the more advanced courses; thus a serious problem is indicated if beginning students continue in their mathematical training. Also many of the large universities have employed many graduate assistants, and have been forced to have freshman and sophomore mathematics taught to a large extent by these graduate assistants. Some are excellent teachers, while others are of questionable ability, and under ordinary circumstances would not be given teaching assignments. Almost all institutions report that classes are much larger. In fact, the shortage of classrooms, along with the inadequacy of essential staff, has forced several schools into handling students in classes of one hundred or more. It is common opinion that the general quality of undergraduate instruction has deteriorated in the past few years.

Several methods are being used to remedy this situation and to raise the effectiveness of the teaching of mathematics. Some schools are planning the courses ahead in detail, and along with this, employing standardized examinations. Others are sectioning their students according to the background and previous records of the students, then distributing the experienced and proven teachers to best advantage. In many of the larger schools graduate fellows and inexperienced instructors have met regularly with a supervising professor to map out methods of presentation; this has led frequently to a plan for close supervision of course-content, methods of presentation, and departmental testing. Other aids such as visiting of classes and conferences with individual teachers have been employed.

There were no reports of undergraduate students being denied admission to mathematics courses. This is indeed a fine record. As soon as more competent teachers are available, it is generally believed that undergraduate teaching will measure up again with that in 1940.

New teachers must come from those students now in our graduate schools. Here the picture is very favorable, although it must be remembered that many graduate students enrolled at the present time will obtain their degrees, and find their way into research with the government, or in industry, rather than make teaching their life profession. Also the question naturally arises whether the graduate schools are able, with the huge increase in the size of enrollment, and the shortage of capable personnel, to give competent training to their graduate students.

At present the registration in mathematics in the graduate schools is at least double any enrollment prior to the war, and all indications are that it will continue to increase rapidly in the years just ahead. This is partly a result of the accumulation of many men who had planned to go directly from college to graduate school, but unfortunately had their training interrupted, after only

the undergraduate course was completed. This group, plus many who finished their undergraduate courses this past spring and were offered attractive assistantships, have filled the graduate schools to overflowing. Although some schools have increased their graduate registration only slightly, others have felt a tremendous increase. For example, one midwestern university in 1940–41 had a graduate enrollment in mathematics of 243 course-registrants while in 1946–47 there were 1344 course-registrants.

No definite generalization can be made with regard to any significant change in the quality of graduate teaching as reported by representatives of the universities. Typical replies were the following from three different institutions:

> On the graduate level, instruction is perhaps superior to what we had before the war, but if enrollments should increase, the manpower situation will become hopeless.
>
> Graduate instruction has been adversely affected by the loss during the war period of full time members of the staff, which have not, up to the present time, been repaired by satisfactory replacements (not to mention additions).
>
> At the graduate level the regular staff members of prewar status are seriously overloaded with the probability that graduate teaching will ultimately suffer if relief does not become available. This problem will become increasingly acute as graduate students reach the stage of thesis research in increasing numbers.

3. Space and equipment. Existing needs for space and equipment are more pertinent in the teaching of the experimental sciences, although to some extent they are also factors in the effectiveness of mathematics teaching. Office space seems to be definitely inadequate, with the result that informal conferences between faculty members and among faculty members and students are difficult. Also the present lack of sufficient classrooms adequately equipped with blackboards has an important bearing on teaching; because of this shortage in several universities, classes are being held continuously from 8:00 A.M. until 10:00 P.M. Nor does there seem to be much hope for any relief in the future. One university, for example, states that in a new building for its College of Liberal Arts the space assigned to mathematics will be fifty percent less than the department occupied twenty years ago, and this in spite of expansion of both the student body and the teaching staff.

The availability of research journals and reference treatises is essential to work in mathematics. Most institutions which reported have been able to maintain their libraries except for some books and periodicals from foreign countries; in many cases these latter items are now being obtained. The need for funds, of course, is ever present.

Other than library facilities, the lack of equipment affects a mathematics department mainly in its courses in statistics. Due to the increase in importance and emphasis on statistics, and the resulting large enrollments, problems in teaching have arisen. Computing machines, although again available, are expensive. Many colleges obtained some computing machines as surplus property, but a majority of these machines lacked the multiplication and division features and hence were not well adapted for use in statistics classes. The situation in one midwestern university, although much more desperate than at most institutions,

might be presented. In the second term, 1942–43, there were approximately 100 registrations in statistics, while in the fall term, 1946–47, the registration for the same course had grown to over 1200. Twenty-two computing machines were available for student use in 1943, and there had been no increase when the report was made. The former statistics laboratory has been taken over by the psychology department, and thus the mathematicians are now limited to a computing room which will hold at most sixteen students. To offset this, it should be stated that the large majority of replies were negative to the question, "Is the quantity and state of repair of your instructional equipment a major limiting factor in determining the effectiveness of teaching in your department at present?"

4. Availability of teachers. In the majority of universities, the undergraduate staff of teachers is much larger and less competent than in 1940. Moreover, the size of the staff has not increased in proportion to the increase in the student body. The problem of adequate staff is made more difficult because of the unpredictability of future needs. Many of the new teachers in colleges and universities are former high school teachers. Also, in certain localities, men from research laboratories have given excellent assistance by teaching of evening classes.

A common response pertaining to the size of the permanent and graduate staff seemed to be, "Our present staff is essentially the same in number and competence as before the war." The situation is not entirely a happy one, however. One department chairman of a midwestern university commented,

> Our present staff numbers about the same as the pre-war staff, but it does not compare quite so well in competence. War work took a good portion of a carefully built-up pre-war staff. Even in normal times such a staff can not be rebuilt over night. Competition of other institutions has considerably affected our staff. Top men are being "pirated" . . . .

From an Eastern university came the comment:

> In 1940 we had thirteen full-time staff members of our department, all with the Ph.D. degree in mathematics, and about eight or nine part-time teaching assistants. This year (1947) we have nine full-time staff members and thirty part-time teaching assistants. We are trying to improve this situation for next year; but the competition for well trained men is keen, and we are finding it difficult to enlarge our full time staff.

These indicate that competition from other universities is felt. Also the competition with industry and government is important. For example, another head of one of the midwestern institutions stated,

> Government competition has taken away the head of our statistics department, and to date we have been unable to replace him. . . . Two of our good mathematics teachers who went to work with the Bell Aircraft Company during the war have not returned because we are not able to offer them a salary commensurable with what they were receiving in industry.

The economic law of supply and demand is fundamental throughout the picture. During the war young students in mathematics were unable to pursue their graduate studies under normal conditions. A large portion of the graduate students were drafted or drawn off into war jobs which offered them no oppor-

tunity for completing their graduate training. Thus in the war years, the number of Ph.D.'s granted in mathematics decreased very rapidly until in 1944 it was only about thirty percent of the normal figure. A corresponding decrease also occurred in the number of Master's degrees granted. Thus, mathematics is now faced with a tremendous shortage of men who have completed their graduate training. When this shortage is considered along with the fact that many colleges and universities are expanding, while some institutions that never required the Ph.D. degree for staff appointments are now trying to build graduate departments, the seriousness of the situation becomes evident. Also the competition for scientific personnel with government and industry, a competition which will be felt for some time, will affect the quality of available staff members at all universities. This is particularly true in the field of mathematical statistics where men who have just obtained the Ph.D. degree are being offered salaries over twice those normally available for men at that stage of accomplishment.

It is generally felt that present salary schedules for science teachers are having and will have in the future an important bearing on the quality of science teaching. If universities can not compete with government and industry for the service of a large portion of the best young scientists available, and consequently these men are lost to the academic world, then the teaching profession will be forced to fill their places with men of inferior quality. This will most certainly have an adverse influence on the training which young scientists receive in the future.

The question was asked: "How effective is science teaching at the graduate level for the training of teachers of science and how might this effectiveness be improved?" Most of the heads of mathematics departments who replied believed that the training given was entirely satisfactory. In fact, only six had any additional comments to make. In some cases the effectiveness of teachers who do research and who must provide leadership in guiding graduate students of science is impaired by too high teaching loads. Also the effectiveness in training future teachers might be enhanced according to some, if more general courses were offered and more use were made of the applications of theory in mathematical and physical problems. Some implied that more training in philosophy and history of mathematics, together with its applications to other sciences, would improve the quality of output. Some insisted that too little attention is paid to the development of teaching skills; in this connection it is suggested the method of practice teaching used for training secondary school teachers might be of some help. In fact, as one man commented (and many would agree, I feel confident),

> There has been practically no training of graduate students for the teaching of mathematics on the college level. I think those departments of mathematics, who turn out the most Ph.D.'s and consequently the largest proportion of college teachers in this subject, should make definite provision for training of graduate students in good teaching habits and methods for their subject.

As to the desirability of having more women teachers of science at the col-

lege and university level, there seems to be a definite hesitancy. Possibly this is caused by the belief stated on several questionnaires that very few women are good enough mathematicians to make a substantial contribution to the training of graduate students. Many institutions, however, would be quite willing to have more women teachers at the college level if they met the scientific qualifications for the position.

5. The influence of the Federal government. At the present time the presence of Federal contracts for research has had little obvious effect on the quality of science teaching in departments of mathematics. In four or five cases the granting of such contracts was reported as beneficial to the effectiveness of teaching, while it was considered harmful by only two.

With regard to policies that educational institutions should follow in the acceptance of Federal contracts for research, including a consideration of the effects of such contracts on teaching, there were several distinctly different replies. Some stated that contracts should be accepted covering fundamental research only, and only when the proposed contracts fit into existing research programs. Others felt that all contracts should be accepted when at all possible, since in the near future there would not be enough stipends available to subsidize graduate students, and research contracts would supply this deficiency. Still other schools felt that such contracts might hamper the universities unduly in the execution of their primary function of training students for teaching and fundamental research. Since this is such a fundamental matter, and one to which most mathematicians have given much thought, I quote the comments of three influential mathematicians, and chairmen of large departments, in this country:

1. I believe that educational institutions in the acceptance of Federal contracts for research should stipulate that in most cases the research men involved should be appointed on a split basis on which about half time should be spent in university teaching and only half time on the contracts. It should also be specified that so far as national safety is not involved, individuals should retain the right of publication. Under these conditions the teaching profession will be enriched by the research activity of the teachers without the customary cost for research purposes of the staff. If scientists devote their entire time to research, the total progress of the research itself may be very great but the benefits to teaching and to the students are likely to be negligible.

2. I certainly do not believe an educational institution should secure a Federal grant, then rob some other institution for men to do the work. This takes men out of teaching where they are so much needed at present. The teaching will certainly be poorer and the rate of training men will be reduced. I believe too much emphasis is being put on research *at the present time*. For a few years the emphasis should be put on teaching and training young men. Unless this is done, the situation will get worse instead of better. Taking men from the teaching staff of our universities is "killing the goose that laid the golden egg." Why doesn't the Federal government do something to help the universities obtain trained staffs in the immediate emergency of increased enrollments? These large enrollments are due largely to the action of the government in sending the veterans to college.

3. In the acceptance of Federal contracts for research, consideration should be given to the following:

(a) Is the research significant in the judgment of men who are competent in the field?

(b) Would the work be done if a Federal contract were not forthcoming?

(c) Is the university in question avoiding its proper obligations? If a university does not encourage and support research, it is no longer a university.
(d) Would the granting of such a contract accentuate the unfortunate separation of education and research?
(e) Under Federal contracts, to what extent would research be directed into specific channels, which may or may not be desirable?

It is interesting to note that several universities, requested by the Federal government to provide assistance in training scientific employees for them, did so willingly, and reported that in no case did it affect the quality of teaching. For example, graduate instruction was provided for qualified personnel at Fort Monmouth; instruction was given to members of the Ames Aeronautical Laboratory, Moffett Field; courses were set up for employees of the United States Navy at the Underwater Sound Laboratory in New London; and, of course, several schools are participating in the NROTC program, thereby offering a place for the training of "junior mathematicians."

6. The role of research. Only a small number of institutions stated that there were any significant changes in the type or amount of research now being carried on. If a change was reported, it was either an increase in applied research, or an increase in both applied and fundamental. The reasons given in case of an increased amount of research were either (1) an increase of staff, (2) an increase of university funds for research, or (3) a contract or project with some branch of the Government.

The comments immediately above were the result of the question, "Has the role of research in your department changed significantly since before the war? Are you, for example, doing more or less applied research as distinguished from fundamental research?" The idea was expressed frequently in this connection that one must not place too much emphasis on the distinction between pure and applied mathematics; mathematics, is indivisible, and one must not emphasize the diversity but rather its unity. Research must always remain an excursion into the unknown, with truth alone as its objective.

With the present overcrowded enrollments in all universities, it is gratifying to know that over three-fifths of the reports indicated that staff members had adequate time and opportunity for research and study.

7. General. The last section of the request for information concerning the effectiveness of science teaching should be considered in two parts. In the first, the following questions were asked: "Is sufficient attention given, particularly in the undergraduate curriculum, to

(a) the meaning and significance of the scientific method?
(b) the recent advances in the field of science being studied?
(c) the fields of science related to the major field of study and the overlapping fields as, for example, biophysics and geochemistry?
(d) the "general education" of the future scientist—training in other fields—training for citizenship?"

Because of the generality of the questions, it seems impossible to summarize the replies. Many who replied answered parts (a), (c) and (d) in the affirmative, though not in emphatic fashion. With respect to point (b), it seemed generally agreed, especially in any undergraduate curriculum, that keeping abreast with the recent advances in mathematics would be an impossibility. It should be stated in this connection, that there appears to exist a definite dissatisfaction with present freshman and sophomore courses. Complete and thorough studies of this undergraduate curriculum are taking place in many institutions.

What is somewhat alarming to the writer is the fact that many mathematicians in this country stated that the above questions were not pertinent to mathematics. It is my personal feeling that insufficient attention has been given to the meaning and significance of the scientific method, that the tendency has been to train students well in particularly narrow fields, but not to give them a broad background in the whole field and in related fields.

The second part of this general section asked for a discussion of the somewhat controversial statement: "This country has, in the past, produced about one-seventh of the Nobel prize winners in science. Do you consider this fact to be a reflection on the quality of science teaching in this country in the past?" Since Nobel prizes are not awarded in mathematics, many declined to comment. Some admitted that this was a reflection on the quality of science teaching, since our educational system was too "practical"; that is, if a future use of knowledge is not in prospect, the knowledge is considered useless. This attitude, they maintained, was destructive of real scientific mentality. The majority contended, however, that the situation with respect to Nobel prize winners was no reflection on science teaching in this country. Several reasons were stated. American educational philosophy is to extend educational opportunity to the many, and not to the intensive training of a few. Also, it is asserted, there seems to be in this country a smaller proportion of time and energy which can be given to research, in comparison with that of scientists of other countries. In this connection also the belief was expressed that there has not been sufficiently early recognition of research ability in this country, and opportunity has not been given scientists to carry on research, unhampered by activities necessary to the individual's economic situation.

DENUMERABILITY OF THE RATIONAL NUMBER SYSTEM

L. S. JOHNSTON, University of Detroit

It has long been known that the set of rational numbers is denumerable, and several counting systems have been devised. It is proposed in this note to exhibit a counting system which furnishes the solution for each of the following problems:

(i) To find the ordinal number for a given rational; for example, to find the ordinal number of the rational 37/11.

(ii) To find the rational number corresponding to a given ordinal; for example, to find the 593rd rational number.

If

$$0 \leqq \frac{x}{y} < \frac{z}{w}, \quad \text{then} \quad \frac{x}{y} < \frac{x+z}{y+w} < \frac{z}{w},$$

whence also

$$\frac{x}{y} < \frac{2x+z}{2y+w} < \frac{x+z}{y+w} < \frac{x+2z}{y+2w} < \frac{z}{w}.$$

This is of course the density property of the rational number system, and between any two fractions we may insert a fraction of intermediate value as shown in the last display. We shall use $[m, n]$ for the fraction $(mx+nz)/(my+nw)$. Then we have the array

$$\begin{array}{cccccccccc}
[1,0] & [0,1] & & & & & & & & \\
[1,0] & [1,1] & [0,1] & & & & & & & \\
[1,0] & [2,1] & [1,1] & [1,2] & [0,1] & & & & & \\
[1,0] & [3,1] & [2,1] & [3,2] & [1,1] & [2,3] & [1,2] & [1,3] & [0,1] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

where we have omitted for typographical convenience the $<$ sign between two successive terms. This array, which we shall call S, has an infinite number of rows, but a finite number of elements in each row. For $x=0$, $y=z=w=1$, $m=0, 1, 2, 3, \cdots$, $n=1, 2, 3, \cdots$, $[m, n]$ is the proper fraction $n/(m+n)$; if m and n are taken always relatively prime, as they are in this note, then $[m, n]$ is a proper fraction in its lowest terms.

We shall denote by $s(i, j)$ the term in the ith row and the jth column of S, and shall mean by $s(i, j)+s(i, j+1)$ the vector addition $[m, n]+[p, q] = [m+p, n+q]$. Inspection of S, in the light of its construction, discloses the following properties

(a) $$s(i, j) = s(i+1, 2j-1),$$

(b) $$s(i, j) + s(i, j+1) = s(i+1, 2j).$$

From (a) it is evident that an element appearing in S at all appears an infinite number of times thereafter, but always in an odd-numbered column after its first appearance. If, therefore, an element $[m, n]$ appears in an even-numbered column of S, it can not have previously appeared in any column of S, nor can it ever appear again in any even-numbered column.

Now delete from S every term of the first column except the first element, $s(1, 1)$, every term in odd numbered columns after the first, and also the first element, $s(1, 2)$, in the second column. Then the residual array includes zero and every proper fraction (note that it has already been stated that every $[m, n]$ will be shown to appear in S) but does not include unity. Denoting by $R(0)$ the set of all proper fractions together with zero, every number of $R(0)$ appears in S.

We shall call the elements of the residual array described above counted elements of S. A little inspection shows that the number of counted elements in the first i rows of S is 2^{i-1}, and therefore that the element $s(i, 2j)$ is the $(2^{i-2}+j)$th counted element of S. We shall define $Q(m, n)=2^{i-2}+j$, where i and j depend for their values upon some method of computing them from m and n. This number $Q(m, n)$ is the ordinal number of the proper fraction $n/(m+n)$ in the set $R(0)$.

To compute i and j for a given m and n, we note first that

$$\begin{aligned}[m, n] &= m \times [1, 0] + n \times [0, 1] \\ &= m \times s(1, 1) + n \times s(1, 2).\end{aligned}$$

To proceed beyond this point it is expedient to exhibit a specific case, the general method being obvious thereafter. Consider the fraction 4/11, or

$$\begin{aligned}[7, 4] &= 7s(1, 1) + 4s(1, 2) \\ &= 3s(1, 1) + 4s(1, 1) + 4s(1, 2).\end{aligned}$$

But from property (b) of S, we have $4s(1, 1)+4s(1, 2)=4s(2, 2)$; from property (a) we have $3s(1, 1)=3s(2, 1)$. Then we have

$$\begin{aligned}[7, 4] &= 7s(1, 1) + 4s(1, 2) \\ &= 3s(2, 1) + 4s(2, 2) \\ &= 3s(3, 2) + s(3, 3) \\ &= 2s(4, 3) + s(4, 4) \\ &= s(5, 5) + s(5, 6) \\ &= s(6, 10) + 0\end{aligned}$$

showing that $[7, 4]=s(6, 10)$, whence $Q(7, 4)=2^{6-2}+5=21$. Hence 4/11 is, in our numbering system, the 21st rational fraction, counting zero as the first.

It will be noted that we have developed above an algorithm quite suggestive of the Euclidean Algorithm on Highest Common Factor. It might also be called a "Method of Finite Descent."

The algorithm can be constructed more simply. Write the columns

(A)	(B)
7,4	1
3,4	1
3,1	2
2,1	3
1,1	5
1,0	10

Column (A) is the set of coefficients of the terms $s(i, j)$ in the first column on the right hand side of the algorithm, and column (B) is the set of j's in the first s of each pair in the algorithm. The sequence of elements in (A) is quite apparent on inspection, and needs no elaboration.

Now let $A(k)$ be the kth element in (A), and let $B(k)$ be the kth element in (B). If the second number in $A(k+1)$ is the same as the second number in $A(k)$, we say that $A(k+1)$ shows a permanence; if the second number in $A(k+1)$ is different from the second number in $A(k)$, we say that $A(k+1)$ shows a variation. Thus $A(1)=7, 4$; $A(2)=3, 4$; and $A(3)=3, 1$. $A(2)$ then shows a permanence, while $A(3)$ shows a variation. (It is quite evident that we are borrowing an idea from Descartes here). To construct column (B), we note that the first element of (B) is always unity. If $A(k+1)$ shows a permanence, then $B(k+1)=2\times B(k)-1$, while if $A(k+1)$ shows a variation, then $B(k+1)=2\times B(k)$. Thus it will be noted that $A(3)$ shows a variation, and $B(3)=2\times B(2)$. But $A(4)$ shows a permanence, whence $B(4)=2\times B(3)-1$. This gives column (B) quite readily. It will be noted that the last two elements of (A) are always 1, 1 and 1, 0, which shows a variation in the last element. This guarantees that the last element of (B) is always even, as has been predicted. It is also quite evident from the nature of the algorithm that any given number pair $[m, n]$ reduces, by a finite number of steps, to the final forms in (A). This bears out a prediction already mentioned twice, namely, every proper rational fraction does appear in S. The number of elements in (A), and of course in (B), is the required value of i, and the last element in (B) is the required value of $2j$. Some emphasis upon the algorithm is worth while. Consider the number pairs [23, 7] and [7, 23].

For the pair [23, 7]:

(A)	(B)
23,7	1
16,7	1
9,7	1
2,7	1
2,5	2
2,3	4
2,1	8
1,1	15
1,0	30

For the pair [7, 23]:

(A)	(B)
7,23	1
7,16	2
7, 9	4
7, 2	8
5, 2	15
3, 2	29
1, 2	57
1, 1	114
1, 0	228

whence $[23, 7] = s(9, 30)$ whence $[7, 23] = s(9, 228)$.

This is an interesting property of S. It is evident that S must exhibit $[m, n]$ and $[n, m]$ in the same row. Furthermore, since there are, in the ith row of S, $2^{i-1}+1$ elements, it follows that if $[m, n] = s(i, 2j)$, then $[n, m] = s(i, 2^{i-1}-2j+2)$. In our problem above we have $30+228 = 2^{9-1}+2$. This is often a convenient check on the computations in (B).

The algorithm can be condensed still further. Inspection shows that if $A(k)$ is followed by a sequence of r variations, then $B(k+r) = 2^r \times B(k)$, while if $A(k)$ is followed by a sequence of r permanences, then $B(k+r) = 2^r \times B(k) - (2^r - 1)$. To apply this, consider again the pair $[23, 7]$. Here $[23, 7]$ is followed by three permanences, reducing to $[2, 7]$. Then $B(4) = 1$. $[2, 7]$ is reduced by three variations to $[2, 1]$, giving $B(7) = 2^3 \times B(4) = 8$. Then $[2, 1]$ is reduced by one permanence to $[1, 1]$, giving $B(8) = 15$. Finally, $[1, 1]$ is reduced to $[1, 0]$ by one variation, giving $B(9) = 30$.

We may set up a tabular arrangement for the procedure above.

(A)		(B)
23, 7		$B(1) = 1$
	$3P$	
2, 7		$B(4) = 1$
	$3V$	
2, 1		$B(7) = 8$
	$1P$	
1, 1		$B(8) = 15$
	$1V$	
1, 0		$B(9) = 30$

Similarly for $[7, 23]$

(A)		(B)
7, 23		$B(1) = 1$
	$3V$	
7, 2		$B(4) = 8$
	$3P$	
1, 2		$B(7) = 2^3 \times 8 - (2^3 - 1) = 57$
	$2V$	
1, 0		$B(9) = 2^2 \times 57 = 228.$

The similarity to the Euclidean Algorithm is even more conspicuous here.

The algorithm is reversible. Consider, for example, the problem of finding $[m, n]$ for $s(6, 10)$. Here write (B) first:

10
5
3
2
1
1

continuing the set of unit elements at the end until six elements appear, since $i=6$. Then write alongside (A) the column (B), remembering that if an element of (B) is even, then the corresponding element of (A) must have shown a variation, while if an element of (B) is odd, then the corresponding element of (A) must have shown a permanence. Then we have

(B)	(A)
10	1, 0
5	1, 1
3	2, 1
2	3, 1
1	3, 4
1	7, 4

whence $s(6, 10) = [7, 4] = 4/11$. Similarly, for $s(9, 228)$ we have

(B)	(A)
228	1, 0
114	1, 1
57	1, 2
29	3, 2
15	5, 2
8	7, 2
4	7, 9
2	7, 16
1	7, 23

whence $s(9, 228) = [7, 23] = 23/30$.

We have shown how to compute $Q(m, n)$ for a given i and j. The procedure here is reversible. For example, let us find $[m, n]$ from $Q(m, n) = 21$. Here we have $2^{i-2}+j=21$. Since j is never zero, we must find the largest integer for i consistent with $2^{i-2}<21$. Then $i=6$, whence $j=5$, and $2j=10$. Then we must find $[m, n]$ such that $[m, n]=s(6, 10)$. This gives, as we have seen, $[m, n] = [7, 4]$. Similarly, to find $[m, n]$ if $Q(m, n)=242$, we have $2^{i-2}<242$, whence $i=9$. Since $2^{i-2}+j=242$, we have $j=114$, $2j=228$. Then we must find $[m, n]$ from $s(9, 228) = [m, n]$, and we have seen that the solution is $[7, 23]$.

We are now prepared to order the whole field of rational numbers. Let $x=p$, $z=p+1$, with the same hypotheses on y, w, m, and n as before. Let $R(p)$, $(p=0, 1, 2, 3, \cdots)$, be the set of positive numbers of the form $p+n/(m+n)$. Let $E(p, Q)$ be the number in $R(p)$ whose ordinal number in $R(0)$ is determined by $Q(m, n)$. It is easy to see from our foregoing remarks that if a proper fraction bears the ordinal number $Q(m, n)$ in the set $R(0)$, then $p+[m, n]$ bears the ordinal number $Q(m, n)$ in the set $R(p)$. Thus $4/11$ is the 21st number in the set $R(0)$, $15/11$ is the 21st number in $R(1)$, and so on. Then we have $E(0, 21)=4/11$, $E(1, 21)=15/11$, and so on. Now form the array:

$$
\begin{array}{cccc}
E(0,1) & E(0,2) & E(0,3) & E(0,4)\cdots \\
E(1,1) & E(1,2) & E(1,3) & E(1,4)\cdots \\
E(2,1) & E(2,2) & E(2,3) & E(2,4)\cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & E(p,Q) & \cdots
\end{array}
$$

Counting the elements of this array in the familiar Cantor order, it is easy to see that the element $E(p, Q)$ is the $[(p+Q)(p+Q-1)/2+Q]$th term in the array. If we were interested only in positive rationals, this last expression would be the ordinal number of $E(p, Q)$ in the set of all positive rationals, including 0. To extend the ordering to negative rationals, we enlarge our counting system and give to every positive rational, including zero, an odd ordinal number and to each negative rational an even ordinal. This can be done easily by defining our ordinal numbers in the following fashion: For a positive rational of the form $p+[m, n]$, the ordinal number is given by

$$N = (p + Q)(p + Q - 1) + 2Q - 1,$$

and for a negative rational of the form $-(p+[m, n])$ the ordinal number is

$$N = (p + Q)(p + Q - 1) + 2Q - 2.$$

We shall carry through a complete solution of the two problems stated at the beginning of the note.

(i) To find the ordinal number of 37/11.

Here $p+[m, n]=3+[7, 4]$. We have already found that $Q(7, 4)=21$. Hence we have $N=24\times 23+42-1=593$; from our extension to negative rationals, we have, for $-37/11$, the ordinal number 592.

(ii) To find the 593rd rational number.

Since the ordinal number is odd, our result will be a positive rational. We must have $(p+Q)(p+Q-1)+2Q-1=593$. Since Q is never zero, we must have also $(p+Q)(p+Q-1)<594$, where $p+Q$ and $p+Q-1$ are the two largest consecutive integers satisfying the inequality. Inspection gives $p+Q=24$, whence $Q=21$, $p=3$. But we have already shown that if $Q(m, n)=21$, then $[m, n]=[7, 4]=4/11$, whence the required number is $3+4/11=37/11$.

We exhibit one more example. Let us find the 6630th rational number. Here we have $(p+Q)(p+Q-1)+2Q-2=6630$, or $(p+Q)(p+Q-1)<6632$. This gives $p+Q=81$, whence $p=5$, $Q=76$. Retracing the algorithm for m and n from $Q(m, n)$, we find that the required element is $s(8, 24)=[18, 7]$. The required number is then $-(5+7/25)=-132/25$.

ON THE TWO-ANGLE POLE OF A LINE TO A TRIANGLE

R. GOORMAGHTIGH, Bruges, Belgium

1. Introduction. In a recent paper [1], I mentioned a theorem generalizing those ideas that lead to the notion of the orthopole and the isopole; it suggests the consideration of a point which is related to a triangle $A_1A_2A_3$, a straight line l, and two angles θ and ϕ, and which therefore may be called the *two-angle pole* of the line to the triangle.

In this paper [1], I gave several properties of the two-angle pole, generalizing well known theorems on the orthopole and the isopole. We will now consider the two-angle pole in connection with subjects which have been discussed previously in the MONTHLY, namely, theorems related to pairs of triangles, and to the orthopolar line of a straight line as to a quadrilateral.

2. The two-angle pole. The theorem which leads to the notion of the two-angle pole is the following one.

THEOREM. *Parallels through the vertices of a triangle $A_1A_2A_3$ cut the line l at P_1, P_2, P_3, forming with l an angle θ; the lines through P_1, P_2, P_3, forming with the corresponding sides of $A_1A_2A_3$ an angle ϕ, form a triangle $Q_1Q_2Q_3$ similar to $A_1A_2A_3$, the ratio ρ being*

$$\rho = \frac{\sin(\theta + \phi)}{\sin \theta}.$$

When $\theta+\phi=\pi$, we find the well known isopole theorem.

The center ω of the circle $Q_1Q_2Q_3$ will be called the pole of l as to $A_1A_2A_3$ for the angles θ, ϕ.

Consider complex coördinates, the base-circle being the circumcircle Γ of $A_1A_2A_3$, the origin being O, and the unit point Ω; let t_1, t_2, t_3 be the coördinates of A_1, A_2, A_3, and denote by s_1 and s_3 the expressions

$$s_1 = t_1 + t_2 + t_3, \qquad s_3 = t_1t_2t_3.$$

Denote by $\bar{a}$ the conjugate to a.

If

$$\lambda = e^{2i\theta}, \qquad \mu = e^{2i\phi},$$

the two-angle pole of the straight line l, having the equation

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1,$$

is easily found to have for coördinate

$$x = \frac{s_1 - \lambda a - \mu \dfrac{\bar{a}}{a} s_3}{1 - \lambda}.$$

3. Pairs of triangles. The following theorem has been considered several times [2] and also in the MONTHLY [3]:

THEOREM. *If two triangles are inscribed in the same circle, the orthopoles of the sides of each of them as to the other triangle are six points on a circle having as center the midpoint of the segment between the orthocenters of the two triangles.*

Let τ_1, τ_2, τ_3 be the coördinates of the vertices of a second triangle $\beta_1\beta_2\beta_3$, and denote by σ_1 and σ_3 the expressions

$$\sigma_1 = \tau_1 + \tau_2 + \tau_3, \qquad \sigma_3 = \tau_1\tau_2\tau_3.$$

The pole for the angles θ, ϕ of $\beta_2\beta_3$ as to $A_1A_2A_3$ is

$$\frac{s_1 - \lambda(\tau_2 + \tau_3) - \dfrac{\mu s_3}{\tau_2\tau_3}}{1 - \lambda}$$

or

$$\frac{s_1 - \lambda\sigma_1}{1 - \lambda} + \frac{\left(\lambda - \dfrac{\mu s_3}{\sigma_3}\right)\tau_1}{1 - \lambda}.$$

Hence the circle C passing through the two-angle poles of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ has for its center

$$\frac{s_1 - \lambda\sigma_1}{1 - \lambda},$$

and the square of its radius is

$$\rho^2 = \frac{(\lambda\sigma_3 - \mu s_3)^2}{\mu s_3\, \sigma_3(1 - \lambda)^2}.$$

But the pole for the angles $-\theta, -\phi$ of A_2A_3 as to $B_1B_2B_3$ is

$$\frac{\sigma_1 - \dfrac{t_2 + t_3}{\lambda} - \dfrac{\sigma_3}{\mu t_2 t_3}}{1 - \lambda^{-1}}$$

or

$$\frac{\sigma_1 - \dfrac{s_1}{\lambda}}{1 - \lambda^{-1}} + \frac{\left(\dfrac{1}{\lambda} - \dfrac{\sigma_3}{\mu s_3}\right)t_1}{1 - \lambda^{-1}};$$

this shows that the pole under consideration is also on the circle C.

If two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are inscribed in the same circle, the poles

for the angles θ, ϕ of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and the poles for the angles $-\theta, -\phi$ of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ are on the same circle.

The center of the circle is the point of the axis of the segment between the orthocenters of the triangles from which that segment subtends an angle 2θ.

It may be recalled that if ψ is the *angle associated* with* the two triangles $A_1A_2A_3$, $B_1B_2B_3$ [4], we have

$$e^{i\psi} = \frac{\sigma_3}{s_3}.$$

Hence

$$\rho = \frac{\sin(\theta - \phi - \psi/2)}{\sin\theta}.$$

When $\lambda/\mu = s_3/\sigma_3$, that is, when $\phi = \theta - \psi/2$, it follows that $\rho = 0$.

If ψ is the angle associated with two triangles $A_1A_2A_3$ and $B_1B_2B_3$, which are inscribed in the same circle, the poles for the angles θ, $\theta - \psi/2$ of the sides of $B_1B_2B_3$ as to $A_1A_2A_3$ and the poles for the angles $-\theta$, $-\theta + \psi/2$ of the sides of $A_1A_2A_3$ as to $B_1B_2B_3$ all coincide with the point on the axis of the segment joining the orthocenters of $A_1A_2A_3$ and $B_1B_2B_3$ from which that segment subtends an angle 2θ.

When the two given triangles have a common mean, that is, when $s_3 = \sigma_3$,

$$\rho = \frac{\sin(\theta - \phi)}{\sin\theta}.$$

4. The two-angle polar line in the cyclic quadrilateral. It is well known that the orthopoles of a straight line as to the four triangles having for vertices three out of four given points are on a straight line [5], [6]. The same property may be extended to the isopoles of the line for a given angle [7].

We will now obtain a similar property about two-angle poles, but in the particular case when the four given points are on the same circle.†

Let A_4 be the point on Γ having as coördinate t_4, and denote by S_1, S_3, S_4 the expressions

$$S_1 = t_1 + t_2 + t_3 + t_4,$$
$$S_3 = t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3,$$
$$S_4 = t_1t_2t_3t_4.$$

Then the coördinate to the two-angle pole of l as to $A_1A_2A_3$ is

$$x = \frac{S_1 - \lambda a - t_4 - \dfrac{\mu\bar{a}S_4}{at_4}}{1 - \lambda},$$

* The angle ψ is twice the constant angle formed by the Simson lines of any point on the circumcircle as to the two triangles.

† The property does *not* exist for two-angle poles in the case of *any* four given points.

and, inasmuch as

$$\bar{x} = \frac{-\lambda\bar{S}_1 + \bar{a} + \dfrac{\lambda}{t_4} + \dfrac{\lambda a t_4}{\mu\bar{a}S_4}}{1-\lambda},$$

elimination of t_4 shows that the point under consideration is on the line

$$x + \frac{\mu\bar{a}S_4\bar{x}}{\lambda a} = \frac{S_1 - \lambda a - \dfrac{\mu\bar{a}S_3}{a} + \dfrac{\mu\bar{a}^2S_4}{\lambda a}}{1-\lambda}.$$

The two-angle poles of a straight line l as to the four triangles formed by three of the vertices of a cyclic quadrilateral are on a straight line L.

We call it the two-angle polar line of l as to the quadrilateral.

If $O\Omega$ is parallel to one of the bisectors of the angles formed by the pairs of opposite sides or diagonals of the quadrilateral, $S_4=1$ and the last equation becomes

$$x + \frac{\mu\bar{a}\bar{x}}{\lambda a} = \frac{S_1 - \lambda a - \dfrac{\mu\bar{a}S_3}{a} + \dfrac{\mu\bar{a}^2}{\lambda a}}{1-\lambda}.$$

The two-angle polar line of l as to a cyclic quadrilateral forms an angle $\phi-\theta$ with the symmetric of l as to the bisectors of the angles formed by the pairs of opposite sides or diagonals of the quadrilateral.

Furthermore, L passes through the point

$$\frac{S_1 - \lambda a}{1-\lambda}.$$

But S_1 is the so-called *orthocenter* of the quadrilateral (point obtained by producing three times its length the segment between O and the center of gravity of the vertices), and a is the image of O in l.

The two-angle polar line passes through the point T of the axis of the segment joining the orthocenter to the image of O in l, and such that the segment subtends at T an angle 2θ.

When the angle θ is a constant and ϕ variable, L passes through the fixed point T.

5. Some envelopes. If θ and ϕ being constant, l passes through a fixed point having as coördinate c, then a is $c-\tau\bar{c}$, where τ is a turn, and the equation to L becomes, $O\Omega$ having still the same position,

$$x - \frac{\mu\bar{x}}{\lambda\tau} = \frac{S_1 - \lambda c - \lambda\tau\bar{c} + \dfrac{\mu(S_3 - \bar{c}/\lambda)}{\tau} + \dfrac{\mu c}{\lambda\tau^2}}{1-\lambda}.$$

Hence the coördinate of the contact point of L with its envelope is

$$x = \frac{S_1 - \lambda c + 2\bar{c}\lambda\tau + \dfrac{\mu c}{\lambda\tau^2}}{1 - \lambda}.$$

When l passes through a fixed point, L envelopes a deltoid.

This is an extension of a theorem given by McBrien [6] for the orthopolar line.

If, θ and ϕ being constant, l envelopes a circle concentric to Γ, it follows that a is $k\tau$, where k is a constant and real and τ is a turn; moreover, the equation to L becomes

$$x + \frac{\mu\bar{x}}{\lambda\tau^2} = \frac{S_1 - \lambda k\tau - \dfrac{\mu S_3}{\tau^2} + \dfrac{\mu k}{\lambda\tau^3}}{1 - \lambda}.$$

The point where L touches its envelope is

$$\frac{2S_1 - \dfrac{\mu k}{\lambda\tau^3} - 3\lambda k\tau}{2(1 - \lambda)}.$$

When l envelopes a circle concentric to Γ, L envelopes an astroid.

This is also a generalization of a theorem on the orthopolar line given by McBrien [6].

When $A_1A_2A_3A_4$ remains inscribed in Γ and circumscribed to a conic, then we have the following relations of the form:

$$S_1 = p + qS_4, \qquad S_3 = \bar{q} + \bar{p}S_4,$$

S_4 being now variable and p and q being constants [8].

So we derive from the equation to L the following property:

When a variable quadrilateral remains inscribed in a circle and circumscribed to a conic, the two-angle polar line of a fixed line as to the quadrilateral passes through a fixed point.

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THE BEGINNINGS OF ANALYTIC GEOMETRY IN THREE DIMENSIONS

J. L. COOLIDGE, Harvard University

1. Statement of the problem. In Section 528 of Jowett's translation of Plato's *Republic* we find the writer complaining of "the ludicrous state of solid geometry." His objection is to the fact that in his time, as in our own, geometry was far more developed in the plane than in a space of three or more dimensions. The reasons for this are fundamental. The fact that the data are more simple in the plane than in three or more dimensions, or that fewer independent variables are involved, not only makes the subject easier at the outset, but curiously enough, when we come to the higher reaches, we can introduce powerful techniques in two dimensions which are not available in spaces of greater dimensionality. Naturally Plato knew nothing of all this. His own efforts to improve the situation by studying regular and semi-regular solids produced no great result. What is really curious is that there was a similar lag in the development of analytic geometry. We shall see that a surprisingly long time passed from the publication of Descartes *Géométrie* until something really effective was accomplished for the algebraic geometry of space.

But first we must come to agreement about definitions, for there are different views as to the meaning of the principal terms. It is my own view that what we mean by analytic geometry is essentially the study of loci by means of equations connecting determining coördinates, regardless of the notation employed. Consequently, I hold that, strictly speaking, plane analytic geometry was introduced by the Greeks in their study of the conic sections. In Apollonius, for example, the study of the ellipse is based on the fact that the square of an ordinate is a constant multiple of the distance from its foot to the ends of the diameter in the conjugate direction. Logically then, I should maintain that the beginnings of analytic geometry in three dimensions are to be found in Archimedes' study of conoids and spheroids, where the ellipsoid of revolution, for instance, is the surface whose equation we should write

$$y^2 + z^2 = \frac{b^2}{a^2} x(2a - x).$$

However, if we look into the matter further, we find that Archimedes did not write this equation or anything like it. He used the properties of the circles which were in planes perpendicular to the axis, and of the ellipses in planes through that line.

The delay in extending algebraic methods to three dimensions appears the more surprising when we reflect that, after all, we live in a three-dimensional physical world, and the advantage of having a frame of reference to which we can refer physical objects, *saute aux yeux*. But probably this advantage was not fully understood until Newtonian mechanics made familiar the ideas of instantaneous velocity and acceleration.

2. Descartes and Fermat. In looking for the beginnings of three-dimensional analytic geometry it is natural to turn to the two great Frenchmen who introduced it in the plane. In fact, it is sometimes asserted that Descartes was the first to envisage three-dimensional analytic geometry. This seems to me to be scarcely the case. At the end of the second book of his *Géométrie*, which first appeared in 1637, we find him writing:

> Au reste je n'ai parlé en tout ceci que des lignes courbes qu'on peut décrire sur une superficie plate, mais il est aisé de rapporter ce que j'en ai dit à toutes celles qu'on sauroit imaginer être formées par le mouvement régulier des points de quelque corps dans un espace à trois dimensions.

It is to be noticed first that he is interested in curves, not in surfaces. His method is to project his space curve orthogonally on two mutually perpendicular planes. The space curve is thus the total or partial intersection of two cylinders whose elements are mutually perpendicular. But at this point he makes a curious slip:

> Même si on veut tirer une ligne droite qui coupe cette courbe au point donné à angles droits, il faut seulement tirer deux autres lignes droites dans les deux plans, une en chacun, qui coupent à angles droits les deux courbes qui y sont aux deux points où tombent les perpendiculaires de ce point donné, car ayant élevé deux autres plans, un sur chacune de ces lignes droites qui coupe à angles droits le plan où elle est, on aura l'intersection de ces deux plans pour la ligne droite cherchée.

This says that if we wish a normal to a space curve, we have but to construct normals to the two corresponding plane curves at the corresponding points, and pass through each of these a plane perpendicular to the plane of the curve; but it is easy to show by a particular example that we do not in this way get a normal to the space curve. I can not help wondering whether Descartes did not mean that we may get the tangent to the space curve by erecting planes tangent to the two plane curves, and orthogonal to their planes.

When we speak of Descartes in connection with analytic geometry we naturally think of the man who, excluding the Greeks, was the co-discoverer, Pierre Fermat. Here we meet a curious situation. In 1643, he sent to de Cercavi a manuscript entitled, *Ad locos ad superficiem Isogoge*. Apparently this was first published by Tannery and Henry in the first volume of their edition of Fermat's works which appeared in 1891. In a note at the bottom of page 111 the editors write:

> Fermat, dont le point de départ est le livre d'Archimède *De conoidibus et spheroidibus* a bien reconnu la nécessité de généraliser la notion de la surface cylindrique, aussi que celle des conoïdes (paraboloïdes elliptiques et hyperboloïdes à deux nappes) et sphèroïdes (ellipsoïdes) d'Archimède, qui n'avait traité que les surfaces de révolution, Mais il n'a pas soupçonné l'existence du paraboloïde hyperbolique, ni l'hyperboloïde à une nappe.

This clearly suggests that he was on the look out for quartic surfaces which were not surfaces of revolution. In his third lemma in this communication he writes:

> Si superficies quaepiam planis quotlibet in infinitum secetur, et communis sectio omnium secantium planorum et dictae superficiei sit quandoque circulus quandoque ellipsis et nihil praeteria, superfies illa erit spherois.

This says that if all sections of a surface are either circles or ellipses the surface is a spheroid. Now if this means that every such surface is a spheroid in the Archimedian sense it is clearly wrong, as the general ellipsoid fulfils the condition. If it is a definition of an ellipsoid it is correct. But why does he call it a "lemma"? An exactly similar difficulty occurs in the case of the conoid. A possible explanation is that the word "superficies" means "surface of revolution," but then the note of Tannery and Henry is wrong. I confess that the whole situation is puzzling to me.

The principal use which Fermat makes of these lemmas is to prove propositions such as the following: *Given a number of planes and a point p which moves in such a way that the sum of the squares of its distances to these planes, each measured in a direction which makes a given angle with the normal to the plane, is constant; the locus of p is a spheroid.*

Let us see how this locus will meet a general plane π. The distance from p along a line which makes a fixed angle with the normal is the normal distance divided by the cosine of the angle. The normal distance is the distance to the intersection of the plane with π, multiplied by the sine of the angle of the two planes. Hence the intersection with π is a curve such that the sum of constant multiples of the squares of its distances from given lines is constant, and such a curve was known to be an ellipse. Hence the surface is a spheroid.

I might say, in conclusion, that previously another mathematician might have given a definition of spheroid which justifies Fermat's lemma, but I know of no such before 1643.

3. Wallis. John Wallis in his *Tractatus de sectionibus conicis*, Oxford, 1655, reaches the quadric surfaces in the following very different fashion. He had been deeply influenced by Cavalieri's method of indivisibles. He begins by showing that triangles of equal bases and altitudes are equivalent, that is, they have the same area, because they may be conceived as being composed of the same infinite set of line segments parallel to the bases, but differently placed. The area is not changed even if the lateral sides of the triangle are curved. We might start with any triangle, move one vertex any distance we please parallel to the base, connect the new vertex with one end of the base by any simple continuous curve, then move the infinite number of segments parallel to the base, parallel to themselves to new positions where one end is on this new curve. As the triangle is conceived as the sum of all of these segments, its area has not changed. The segments are conceived as infinitely thin rectangles.

Wallis extends this same process to three dimensions. Let us take as base a plane convex figure which is symmetrical with regard to its center C. Let a line through C meet the base curve in B and S, and let V be a point outside the plane of the base. We obtain a cone by connecting V with every point of the base curve. But Wallis describes the situation differently. Through each point of the axis VC we pass a plane parallel to the base plane, and in it describe a figure similar and similarly placed to the base. Let y be the distance in such a plane between the intersections with VC and VB. The ratio of the area in this

plane to the base area is y^2/CB^2. Wallis next replaces the lines VB, VS by a conic whose tangent at V is parallel to the base. We cut this by a set of planes parallel to the base plane, the ordinate y having the same meaning as before. In each such plane we have a closed figure, similar to the base figure, symmetrical with regard to the intersection with VC, a diameter of the conic; the ratio of the area in this plane to the base area is given by the previous equation. This figure he calls a "pyramidoid" or "conoid," parabolic, elliptic, or hyperbolic, as the case may be. At the end of the article, pages 111 and 112, the words, ellipsoid, paraboloid, and hyperboloid are introduced. In a plane through the axis of the original conic we have a figure similar to this conic, all ordinates being altered in the same ratio. He also says that in a general plane cutting the axis, the section will be a conic, but this is only true when the base is a conic, which he does not explicitly assume. Kötter interpreted Wallis, proceeding much more simply, by saying the surface is generated by a series of similar and similarly placed central conics, in parallel planes, whose axes are the double ordinates of a given conic [1]. This is perfectly true when the base curve is a conic, and we can obtain the general quadric surface in this way, but it is less general than what Wallis actually says, and Kötter apologizes because in the figure for a parabolic pyramidoid the base figure has five sides when he wanted it to have six.

Wallis is particularly interested not in the surface he generates in this way, but in the volumes. The integrations he performs ingeniously. Suppose the original conic is a parabola, leading to

$$y^2 = kx; \qquad \int y^2 dx = k \int x dx.$$

To find this integral, suppose we have an isosceles right triangle on which stands a right prism. The area of the section in a vertical plane at a distance x from the opposite edge can be written so that this integral gives us the volume of the prism, which, however, we know otherwise.

When the original conic is an ellipse, we know from Apollonius that

$$y^2 = \frac{b^2}{a^2} x(2a - x).$$

Suppose then, that we have a tetrahedron whose vertices are $(0, 0, 0)$ $(0, 0, c)(2a, d, 0)(2a, -d, 0)$. The area of a rectangular section determined by the plane, $x =$ a constant, is $cdx(2a-x)/2a^2$, so we get the volume of a pyramidoid or conoid from that of a pyramid. It is worth mentioning that the scheme of finding an integral by comparing the volume of a solid calculated by two different subdivisions is found in Pascal. Fermat, writing in 1643, certainly got no help from Wallis who wrote in 1655.

I must mention one other writer who is supposed to be the first to have introduced Cartesian geometry in three dimensions. I refer to Johann Bernoulli. Here is the opinion of Beniamino Segre:

Poco appreso Joh. Bernoulli, lettere al Leibniz di 4 dicembre 1697, del 26 agosto 1698 a del 6 Febraio 1725, utiliza le coördinate per la determinazione delle geodetiche su certi tipi di superficie [2].

This statement is not absolutely correct. In the first two places mentioned there is no discussion of coördinates. The third is more helpful:

Intelligo per superficiem datam, cujus singula puncta determinatur (sic ut lineae curva puncta) per ordinates tres x, y, z quarum relatio data aequatione ecprimeretur; sunt autem tres illae coordinatae nihil aliud, quam tres tres rectae ex qualibet superficiei curvae puncto perpendiculariter ductae in tres plana positione data, et se mutuo ad angulos rectos secantis [3].

Leibniz' reply appears on the next page:

Doctrinam de aequationibus localibus trium coordinatarum, seu de locis vere solidis, olim aggredi coepi, eorumque intersectiones sive curvas etiam non planas, sed prosequi non vacavit.

This is perfectly clear, but I must point out that the date is 1715, by which time, as we shall see, the method had been publicly explained. In 1728, Johann Bernoulli wrote to Klingsterna:

Sit $AE=x$, $EB=y$ et a puncto B erigi intelligitur recta $Bb=z$ normalis and planam AEB et superficiei curvae occurens in b et detur aequatio quaevis exprimans relationem trium coordinatarum x, y, z qua relatione natura superficiei determinatur [4].

I should also mention in this connection a word by Clairaut:

Je ne crois pas cette matière moins neuve que celle des courbes à double courbure, et je ne scai de connu sur ce sujet, que la façon d'exprimer les surfaces courbes par des equations à trois variables dont j'ai appris qu'il étoit fait mention par occasion dans un mémoire du célèbre M. Bernoulli inseré dans les Actes de Leypsic [5].

In this quotation does "célèbre M. Bernoulli" mean James or John; I find nothing of this sort in James' collected works, nor in anything of John's except what I have already given, and that is not in the Acta Eruditorum.

4. LaHire and Parent. It is time to turn from those who might have invented analytic geometry in three dimensions but didn't, to those who did. First I take Philippe de LaHire. He stated definitely in 1679:

Je considère d'abord pour déterminer un point hors d'un plan, à l'égard d'une ligne droite déterminée sur ce plan, il faut trois conditions; la première est la grandeur LA de la perpendiculaire menée du point L au plan, la seconde la perpendiculaire AB menée du point A à la ligne donnée OB, et la troisième la partie OB de cette ligne comprise entre un de ses points O et le recontre B. C'est pourquoi je fais $OB=x$, $AB=y$, $LA=v$ qui sont trois inconnus.

He takes one specific problem, to find L so that $a+OB=OL$. He sees geometrically

$$OL=\sqrt{x^2+y^2+v^2}; \qquad a+x=\sqrt{x^2+y^2+v^2}$$
$$a^2+2ax=y^2+v^2;$$

then he adds tamely enough

et comme il n'y a pas moyen de trouver d'autres équations pour faire évanouir quelques unes des inconnues, il s'ensuit que le problème est indéterminé [6].

So here we have a capable mathematician with all of three dimensional analytic geometry under his hand, but unable or unwilling to make anything of it save to calculate the distance of a point from the origin, and to note that one single condition will not locate completely a point in three-space.

Very different is the case of LaHire's disagreeable but enterprising fellow countrymen, Antoine Parent, whose paper, *Des affections des superficies*, was read before the French Académie des Sciences on August 23, 1700, and published in his *Essais et recherches de mathématiques et de physique*, Vol. 2, Paris, 1723, pages 181–200.

It is hardly necessary to say that "affection" means equation; Parent is interested in the equations of surfaces. His first general problem is to determine the tangent plane. In his time mathematicians did not bother with the equations of tangents, but with their construction, and in the case of a plane curve, this came from the subtangent, which is the orthogonal projection on the x-axis of as much of the tangent as lies between the point of contact and the intersection with that axis; its value is ydx/dy.

Parent begins with a sphere "pour exemple." He has a perfectly general method, but he begins with a sphere for he knows how to find its equation. Like LaHire he knows how to find the distance of a point from the origin. The radius of the sphere is r, the coördinates of the center are, b, c, a; thus

$$c^2 + y^2 - 2cy + b^2 + x^2 - 2bx + a^2 + z^2 - 2az = r^2.$$

First holding y constant, and calmly disregarding algebraic signs, he has

$$(a - z)dz = (b - x)dx; \qquad \frac{zdx}{dz} = \frac{a - z}{b - x}z.$$

Similarly,

$$\frac{zdy}{dz} = \frac{a - z}{c - y}z.$$

Clearly he has in mind the general formula, which we should write

$$f(x, y, z) = 0, \quad \text{having subtangents} \quad -\frac{f_z}{f_x}z; \quad -\frac{f_z}{f_y}z.$$

Parent next looks for those points on a surface where y takes a maximum or a minimum value. For a surface he takes a conchoid, discussed by L'Hospital, and allows one of the constants to vary; that is, he considers

$$y^2 = \frac{z - x}{x}(b + x)^2.$$

If we treat z as a constant, it follows that

$$2ydy = (b + x)\left[\frac{2(z - x)}{x} - \frac{(b + x)z}{x^2}\right]dx.$$

For an extremal we have $dy=0$; so

$$2x^2 - zx + bz = 0.$$

He does not recognize this as representing a cylinder, but sees that the projection on the xz plane is a hyperbola. By solving this for x, and substituting in the equation of the surface, we get the projection on the yz plane.

Parent's third problem is to find the locus of the points of inflection in the planes, $z=$a constant. If we take x as the independent variable, we have at such points, in the notation of the time,

$$dx = \text{a constant}, \qquad ddy = 0.$$

He needs a surface where y appears explicitly, so he takes

$$y = \frac{z^3}{x^2 + az}.$$

Keeping z constant,

$$dy = \frac{-2xz^3dx}{(x^2 + az)^2},$$

$$ddy = (3x^2 - az)\frac{2z^3dx^2}{(x^2 + az)^3}.$$

For a point of inflection we shall have

$$3x^2 - az = 0.$$

Parent does not go beyond this point; it is evident, however, that he had hold of a very real piece of mathematics for which he should deserve full credit.

5. Alexis Clairaut. I have already spoken of the work of the infant prodigy, Alexis Clairaut; it is time to speak further of his real contributions to three dimensional analytic geometry, even though he was late in time, 1731. Clairaut was inspired by the work of Descartes, which I have mentioned. He was interested in curves, not in surfaces, but he set up Cartesian coördinates exactly after the manner of LaHire and Parent, though he mentioned neither of these writers. His first theorem is that an equation involving three coördinates is a surface, because the intersection with any plane parallel to a coördinate plane is a curve. An equation of the first degree represents a plane, because the intersections with all planes, $z=$a constant, are straight lines parallel to one another, and the same is true for planes, $y=$a constant. He deals no further with planes but develops the formula for the square of the distance between two given points. He shows how to find the equation of a cone determined by a plane curve and a point outside of its plane, and how, when we have the equations of two surfaces, we may, by elimination, find the equation of the projection of their common curve on one of the coördinate planes.

Clairaut is happier when he passes to the application of the calculus to the study of curves. But here I should say that he does not go beyond first derivatives, or first differences, as he would have called them. The first and most fundamental problem is to find the tangent. The actual length of the tangent from the point of contact to the intersection with the plane $z=0$ is $z\sqrt{dx^2+dy^2+dz^2}/dz$. From this we can easily pass to various sorts of subtangents. I note in passing that the square of dx is dx^2, but the square of x is xx.

Most of the particular problems are not sufficiently interesting to be worth reproducing. The most amusing, perhaps, is that of finding the equation of the curve cut in the base plane $z=0$ by a moving tangent to the space curve. We need the orthogonal projections on the x and y axes of so much of the tangent as runs from the point of contact to the base plane. These we get from the previous formulae for subtangents, giving

$$x' = x - \frac{z\,dx}{dz}; \qquad y' = y - \frac{z\,dy}{dz}.$$

From the finite and differential equations of the space curve we may find x, y, dx/dz, dy/dz in terms of z. Eliminating, we have $\phi(x', y')=0$.

The third section of Clairaut's work deals with the applications of the integral calculus to curve theory. He recognizes the distance formula

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

It is naturally hard to find a curve which can be rectified in this way. Here is one:

$$y^2 - 2a^2 = \sqrt[3]{9a^4x^2}; \qquad az = y^2.$$

Thus,

$$x = \frac{(y^2 - 2a^2)^{3/2}}{3a^2}; \qquad dx = \frac{\sqrt{y^2 - 2a^2}\,y\,dy}{a^2}; \qquad dz = \frac{2y\,dy}{a};$$

$$ds = dy\,\frac{y^2 + a^2}{a^2};$$

$$s = \frac{y^3}{3a^2} + y + C.$$

We can find the area of as much of one of the cylinders as is bounded by the curve and its projection on the yz plane from the integral $\int x\sqrt{dy^2+dz^2}$. The volume bounded by the two cylinders and the base plane is given by $dv = yz\,dx$. He finds some other volumes of this sort.

The fourth section gives certain unconnected constructions of space curves. For instance, suppose that we have a compass of fixed opening r and one end is fixed at the point (a, b, c) of a given surface, what curve will the other point trace in the surface? Clearly

$$f(x, y, z) = 0, \qquad (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Here is a more difficult problem: To find the equations of a curve along which a given surface is touched by the tangents through a given point, (a, b, c). We have evidently

$$\frac{dx}{(x-a)} = \frac{dy}{(y-b)} = \frac{dz}{(z-c)}.$$

Here our modern notation gives the answer immediately. Besides the equation of the surface

$$f(x, y, z) = 0,$$

we have

$$f_x dx + f_y dy + f_z dz = 0,$$
$$f_x(x-a) + f_y(y-b) + f_z(z-c) = 0.$$

Clairaut's solution is, naturally, less compact. The lack of a suitable notation for partial derivatives was a serious disadvantage to mathematicians of, Clairaut's time, yet they were recognized by Newton. It is largely for this reason that his book of 119 pages gives the impression of being very long for the actual amount of mathematics involved, but when we remember that the author was only sixteen when his memoir was first presented to the Académie des Sciences, we can but wonder at his precocity.

6. Jakob Hermann. The year after the appearance of Clairaut's "Recherches" another article, dealing with analytic geometry in three dimensions, but clearly independent, first saw the light. The author was Jakob Hermann [7]. His interest was in surfaces rather than in curves. He believed that the subject had been neglected in the past because geometers were deterred by its apparent prolixity. He set up his axes exactly like those of Parent and Clairaut, though neither of these authors is mentioned, and then gave a list of six surfaces he meant to study. The first was

$$ax + by + cz - e^2 = 0.$$

This is shown to be a plane, by means of similar triangles. Next we have

$$z^2 - ax - by = 0.$$

This is a parabolic cylinder. The section in a plane, $z =$ a constant, is a straight line, and all such lines are parallel. In the planes $x =$ a constant or $y =$ a constant, we have a parabola. Let us find an extreme value for z. Here we should have $dz/dx = dz/dy = 0$, but that involves $a = b = 0$, which is not the case, so there is no extreme value which is fairly evident geometrically. To find a tangent plane we seek the sub-tangents, as Parent did; that is,

$$\frac{z dx}{dz} = \frac{2(ax+by)}{a}; \qquad \frac{z dy}{dz} = \frac{2(ax+by)}{b}.$$

The next surface is

$$z^2 - xy = 0.$$

This is a cone as it contains entirely a line connecting the origin with any other point of the surface. Sections perpendicular to the z-axis are rectangular hyperbolas; those in planes perpendicular to the other axes are parabolas. We find something more interesting in his fourth example,

$$z^2 - ax^2 - bxy - cy^2 - ex - fy = 0.$$

Hermann remarks:

Haec Aequatio est ad superficiem alcujus Conoidis, cujus basis exponitur ex aequatione

$$cy^2 + bxy + ax^2 + fy + ex = 0.$$

The wording here is interesting. "Conoidis" does not mean surface of revolution of Archimedes. Hermann deals separately with surfaces of revolution in a subsequent paragraph, and he refers explicitly to the Wallis wedge conoid. It seems certain that he was familiar with Wallis' work, and took his definition of a conoid. He finds the extreme values of z by equating the two partial derivatives to 0; that is,

$$2ax + by + e = 0; \qquad 2cy + bx + f = 0.$$

We solve these for x and y, and substitute in the equation of the surface. He finds the tangent plane from the sub-tangents zdy/dz, zdx/dz. This is so close to Parent's work that one wonders whether he was not familiar with the French writer's mathematics.

Hermann's fifth surface gives something more difficult, namely,

$$az^2 - bxz - cyz + ey^2 = 0.$$

This again he recognizes as giving a cone, and he undertakes the none too easy problem of seeking for the planes of circular section. He takes u and t as rectangular Cartesian coördinates in such a plane, and expresses x, y, z in terms of these and of the constants determining the aspect of the plane; then he substitutes, and writes the conditions for a circle. He arrives finally at an equation of the third degree, as we should expect. The problem was not new. It had been handled in the 75th letter of Descartes, Third series, Ed. 1683, and in No. 441 of L'Hospital's *Traité analytique des sections coniques.*

Hermann's last surface is

$$u^2 - x^2 - y^2 = 0$$

where u is a function of z.

Haec Aequatio ad omnis generis solidum spectat.

He ends with a short study of geodesic curves, but acknowledges that the general problem is beyond him.

We may say that with Hermann the study of analytic geometry in three

dimensions was on a firm basis, and surfaces of the second order familiar objects. For that reason Cajori is clearly wrong when he writes of Euler: "He was the first to discuss the equation of the second degree in three variables, and to classify the surfaces represented by it" [8]. It is however true that he was the first to reduce such equations to canonical forms, in his *Analysin Infinitorum* of 1748. What a long time this is after Descartes' *Géométrie* of 1637, where curves of the second degree were elaborately treated!

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MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

PARAMETRIC SOLUTIONS OF TWO MULTI-DEGREED EQUALITIES

A. GLODEN, Luxembourg

1. Introduction. The notation

$$A_1, A_2, \cdots, A_p \overset{n_1,n_2,\cdots,n_r}{=} B_1, B_2, \cdots, B_q$$

designates a so-called *multi-degreed equality* and means that the sum of the numbers on the left equals the sum of the numbers on the right for each of the $r(n_1, n_2, \cdots, n_r)$ positive integral powers of the numbers.

We shall give here parametric solutions of the two multi-degreed equalities

$$(1) \qquad A_1, A_2, A_3 \overset{2,4}{=} B_1, B_2, B_3, \qquad A_3 \neq A_1 + A_2, \qquad B_3 \neq B_1 + B_2,$$

and

$$(2) \qquad C_1, C_2, \cdots, C_7 \overset{1,2,4,6,8}{=} D_1, D_2, \cdots, D_7.$$

2. Parametric solutions of (1). To obtain solutions for (1), we may proceed as follows. Consider the multi-degreed equality

$$a,\ a \overset{2,4}{=} b,\ c,\ b + c.$$

It implies the following in which $d = b + c$; k and l are arbitrary integers.

$$ak + al,\ bk + cl,\ ck - dl,\ dk - bl \overset{2,4}{=} ak - al,\ bk - cl,\ ck + dl,\ dk + bl$$

We determine the quotient k/l in such a manner that a term on the left equals a term on the right or the opposite number of a term on the right. We shall consider only the solutions in which

$$A_3 \neq A_1 + A_2, \qquad B_3 \neq B_1 + B_2.$$

Here is a parametric solution* of (1):

$$a = m^2 + mn + n^2, \qquad b = m^2 - n^2, \qquad c = 2mn + n^2.$$

Let $(\alpha, \beta, \gamma, \delta, \epsilon)$ be the fourth-degree form $\alpha m^4 + \beta m^3 n + \gamma m^2 n^2 + \delta m n^3 + \epsilon n^4$. We now get the following parametric solutions of (1):

$$\begin{array}{l}
(-1,\ 3,\ 6,\ 7,\ 3),\ (-1,-2,\ 5,\ 6,\ 1),\ (\ 0,\ 4,\ 2,-4,-2) \overset{2,4}{=} \\
(\ 1,\ 1,\ 0,-1,-1),\ (\ 1,-2,\ 1,\ 6,\ 3),\ (\ 0,\ 0,\ 8,\ 8,\ 2) \\
(-2,\ 0,\ 3,\ 5,\ 3),\ (\ 0,\ 4,\ 2,-4,-2),\ (-1,\ 2,\ 7,\ 2,-1) \overset{2,4}{=} \\
(\ 2,\ 0,-4,\ 0,\ 2),\ (\ 0,\ 2,\ 3,\ 3,\ 1),\ (\ 1,-2,\ 1,\ 6,\ 3) \\
(\ 3,\ 5,\ 3,\ 0,-2),\ (\ 2,\ 4,-2,-4,\ 0),\ (-1,\ 2,\ 7,\ 2,-1) \overset{2,4}{=} \\
(\ 1,\ 3,\ 3,\ 2,\ 0),\ (\ 2,\ 0,-4,\ 0,\ 2),\ (\ 3,\ 6,\ 1,-2,\ 1) \\
(\ 1,\ 1,\ 0,-1,-1),\ (\ 2,\ 8,\ 8,\ 0,\ 0),\ (\ 3,\ 6,\ 1,-2,\ 1) \overset{2,4}{=} \\
(\ 1,\ 6,\ 5,-2,-1),\ (\ 2,\ 4,-2,-4,\ 0),\ (\ 3,\ 7,\ 6,\ 3,-1) \\
(\ 0,\ 2,\ 3,\ 3,\ 1),\ (\ 1,\ 6,\ 13,\ 6,\ 1),\ (\ 2,\ 8,\ 8,\ 0,\ 0) \overset{2,4}{=} \\
(\ 0,\ 4,\ 10,\ 4,\ 0),\ (\ 1,\ 6,\ 5,-2,-1),\ (\ 2,\ 8,\ 9,\ 7,\ 1) \\
(\ 0,\ 0,\ 8,\ 8,\ 2),\ (\ 1,\ 3,\ 3,\ 2,\ 0),\ (\ 1,\ 6,\ 13,\ 6,\ 1) \overset{2,4}{=} \\
(\ 0,\ 4,\ 10,\ 4,\ 0),\ (\ 1,\ 2,-5,-6,-1),\ (\ 1,\ 7,\ 9,\ 8,\ 2).
\end{array} \tag{3}$$

Each sextuplet contains for given values of the parameters m, n, 21 distinct numbers of which 15 are repeated twice.

EXAMPLE. We obtain from (3) for $m = 3$, $n = 1$,

$$71,\ 78,\ 112 \overset{2,4}{=} 57,\ 98,\ 104.$$

3. Parametric solutions of (2). We apply now the theorem of Birck: *The multi-degreed equality,*

* A. Gloden, Mehrgradige Gleichungen, 1944, page 90.

$$a_1, a_2, a_3 \overset{2,4}{=} b_1, b_2, b_3 \text{ in which } a_3 \neq a_1 + a_2,\ b_3 \neq b_1 + b_2,$$

implies†

$$(4)\quad \begin{matrix} a_1 + a_2 + a_3,\ -a_1 + a_2 + a_3,\ a_1 - a_2 + a_3,\ a_1 + a_2 - a_3,\ 2b_1,\ 2b_2,\ 2b_3 \\ b_1 + b_2 + b_3,\ -b_1 + b_2 + b_3,\ b_1 - b_2 + b_3,\ b_1 + b_2 - b_3,\ 2a_1,\ 2a_2,\ 2a_3. \end{matrix} \overset{1,2,4,6,8}{=}$$

Each of the 6 parametric solutions of (1) confers a parametric solution of (2). Here is the solution corresponding to (3):

$$(2, -1, 9, 13, 4),\ (0, -3, 9, 15, 6),\ (0, 3, 7, 1, -2),\ (2, -1, -7, -3, 0),$$
$$(-2, 6, 12, 14, 6),\ (0, 8, 4, -8, -4),\ (2, 4, -10, -12, -2) \overset{1,2,4,6,8}{=}$$
$$(-2, 5, 13, 9, 2),\ (0, 1, -1, 5, 4),\ (2, 3, -9, -17, -6),\ (0, 9, 3, -3, 0)$$
$$(2, 2, 0, -2, -2),\ (2, -4, 2, 12, 6),\ (0, 0, 16, 16, 4).$$

The reader may construct the five other parametric solutions of (2).

EXAMPLE. For $m=3$, $n=1$, we have

$$37, 105, 114, 119, 196, 208, 261 \overset{1,2,4,6,8}{=} 51, 63, 142, 145, 156, 224, 259.$$

ON COPRIME VALUES TAKEN BY GIVEN POLYNOMIALS

L. MIRSKY, Sheffield, England

Let r, s be integers such that $2 \leqq r \leqq s$. Let $f_1, \cdots, f_s$ be given polynomials, with integral coefficients, no r of which possess a (non-constant) common factor. An integer n will be called *admissible* if no r of the s integers $f_1(n), \cdots, f_s(n)$ possess a common factor greater than 1.

For any prime p we denote by $D(p) = D(p; r; f_1, \cdots, f_s)$ the number of numbers x among $1, 2, \cdots, p$ for which at least r of the s congruences

$$f_i(x) \equiv O \pmod{p} \qquad (1 \leqq i \leqq s)$$

are satisfied.

We shall prove the following theorem.

THEOREM. *The admissible numbers form precisely* $\prod_{p|K}(p - D(p))$ *residue classes* (mod $\prod_{p|K} p$), *where* K *is a number depending at most upon* $r, s, f_1, \cdots, f_s$. *In particular, therefore, if* $D(p) = p$ *for some* $p \mid K$, *then the set of admissible numbers is empty.*

Proof. Let $1 \leqq i_1 < \cdots < i_r \leqq s$. Since the polynomials $f_{i_1}, \cdots, f_{i_r}$ have no common factor, there exists a least positive integer $K(i_1, \cdots, i_r)$ such that, for suitable polynomials* $g_1, \cdots, g_r$ we have identically

$$f_{i_1}g_1 + \cdots + f_{i_r}g_r = K(i_1, \cdots, i_r). \qquad (1)$$

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† A. Gloden, loc. cit., page 44.

* The polynomials $g_1, \cdots, g_r$ will, of course, generally depend on $i_1, \cdots, i_r$.

We write

$$K = \prod_{1 \le i_1 < \cdots < i_r \le s} K(i_1, \cdots, i_r). \tag{2}$$

If p is any prime, n will be called p-admissible if fewer than r of the s numbers $f_1(n), \cdots, f_s(n)$ are divisible by p. Hence n is admissible if and only if it is p-admissible for every prime p.

Now if p is a divisor of at least r numbers among $f_1(n), \cdots, f_s(n)$, then, by (1) and (2), $p \mid K$. Therefore, if it is not true that $p \mid k$, then every n is p-admissible. Hence n is admissible if and only if it is p-admissible for every $p \mid K$.

The numbers which are p-admissible clearly form precisely $p - D(p)$ residue classes (mod p). But, as is well known, if $m_1, \cdots, m_h$ are coprime in pairs, then the system of congruences

$$n \equiv a_1 \pmod{m_1}, \cdots, n \equiv a_h \pmod{m_h}$$

is soluble, and the solutions form precisely one residue class (mod $m_1 \cdots m_h$). Hence the numbers which are p-admissible for every $p \mid K$, that is, the admissible numbers, form precisely $\prod_{p \mid K}(p - D(p))$ residue classes (mod $\prod_{p \mid K} p$).

REMARKS. (*i*) For $x > 0$ let $N(x)$ denote the number of positive admissible numbers not exceeding x. By the theorem just proved it follows at once that, as $x \to \infty$,

$$N(x) = x \prod_{p \mid K} \left(1 - \frac{D(p)}{p}\right) + O(1), \tag{3}$$

where the O-constant may depend upon $r, s, f_1, \cdots, f_s$.

(ii) Let $k_1, \cdots, k_s$ be given distinct integers, and consider the special case

$$f_i(n) = n + k_i \qquad (1 \le i \le s).$$

A number n is now called admissible if no r among $n+k_1, \cdots, n+k_s$ have a common factor greater than 1, and $D(p)$ is the number of numbers among $1, 2, \cdots, p$ which are congruent (mod p) to at least r among $k_1, \cdots, k_s$. The relation (3) now holds, for instance with

$$K = \prod_{1 \le i < j \le s} |k_i - k_j|.$$

(iii) We may specialize (ii) still further by putting $r = s$. Then $N(x)$ becomes the number of positive integers $n \le x$ such that the highest common factor of $n+k_1, \cdots, n+k_s$ is 1. It is easy to show that

$$N(x) = x \frac{\phi(k)}{k} + O(1),$$

where ϕ denotes Euler's function, and k is the highest common factor of $|k_1 - k_2|, |k_1 - k_3|, \cdots, |k_{s-1} - k_s|$.

I would like to record my indebtedness to Dr. R. Rado for his very valuable suggestions in connection with this note.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

A MODIFICATION OF NEWTON'S METHOD

H. S. WALL, University of Texas

Newton's formula for approximating the roots of an equation $f(x)=0$, namely,

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}, \qquad (p = 0, 1, 2, \cdots) \tag{1}$$

may be modified in the following way. The equation of the parabola through the point $(x_p, f(x_p))$, having the same first and second derivatives at $x=x_p$ as $y=f(x)$, is

$$y = f(x_p) + (x - x_p)f'(x_p) + \tfrac{1}{2}(x - x_p)^2 f''(x_p).$$

Let x_{p+1} be a solution of the equation which results if we put $y=0$. Then

$$x_{p+1} - x_p = \frac{-f(x_p)}{f'(x_p) + \tfrac{1}{2}(x_{p+1} - x_p)f''(x_p)}.$$

If we take $x_{p+1}-x_p = -f(x_p)/f'(x_p)$ in this formula, we then obtain

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p) - \dfrac{f(x_p)f''(x_p)}{2f'(x_p)}}, \qquad (p = 0, 1, 2, \cdots). \tag{2}$$

This is the desired modification of Newton's formula.

In a number of examples to which we have applied formula (2), the convergence of the sequence $\{x_p\}$ is essentially more rapid than in the case of formula (1).

EXAMPLE 1. *To compute the n-th root of a positive number a.* If we take $f(x)=x^n-a$, then Newton's formula is

$$x_{p+1} = \frac{a/x_p^{n-1} + (n-1)x_p}{n}, \qquad (p = 0, 1, 2, \cdots), \tag{3}$$

and formula (2) is

$$x_{p+1} = \frac{(n-1)x_p^n + (n+1)a}{(n+1)x_p^n + (n-1)a} \cdot x_p, \qquad (p = 0, 1, 2, \cdots). \tag{4}$$

The following table gives the approximations to $\sqrt{2}$ obtained from these formulas, starting with $x_0=1$.

	(3)	(4)
x_1	1.500000000	1.4
x_2	1.416666667	1.41421319797
x_3	1.414215686	1.41421356237309504879569008
x_4	1.414213562	

The value of x_3 found by Newton's formula is correct to four decimal places, while the value of x_3 found by formula (4) is correct to nineteen decimal places.

Starting with $x_0=10$, we find that Newton's formula gives for x_4 the value 1.4442, which is correct to one decimal place, while formula (4) gives the approximation $x_4=1.414213562$, which is correct to nine decimal places.

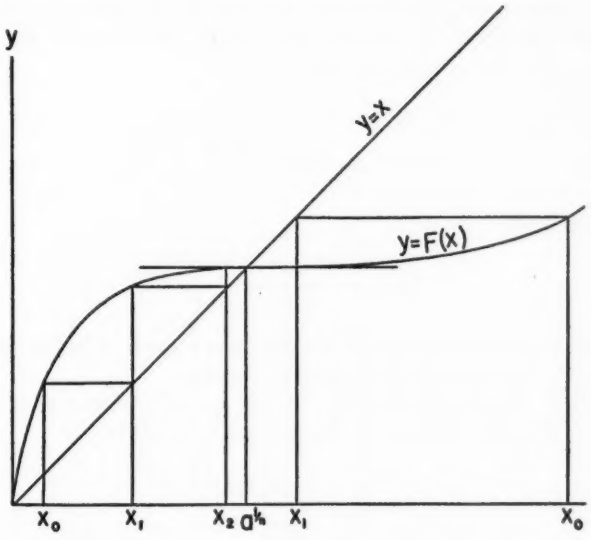

Fig. 1

If $x_0>0$, then the sequence $x_1, x_2, x_3, \cdots$, given recurrently by (4), *converges to $a^{1/n}$, and is an increasing sequence if $x_0<a^{1/n}$ and a decreasing sequence if $x_0>a^{1/n}$. The error committed in taking x_{p+1} as an approximation to $a^{1/n}$ is of the order of the cube of the error committed in taking x_p as an approximation to $a^{1/n}$.*

This can be seen in the following way. The function

$$F(x)=\frac{(n-1)x^{n+1}+(n+1)ax}{(n+1)x^n+(n-1)a}$$

has the derivative

$$F'(x) = \frac{(n^2 - 1)(x^n - a)^2}{[(n+1)x^n + (n-1)a]^2},$$

which vanishes without changing sign at $x = a^{1/n}$, is otherwise positive, and is greater than unity at $x = 0$. Thus, the graph of $y = F(x)$ has a point of inflection with horizontal tangent at $x = a^{1/n}$, where it crosses the line $y = x$. From the figure it is now clear that $x_p \uparrow a^{1/n}$ if $0 < x_0 < a^{1/n}$, and $x_p \downarrow a^{1/n}$ if $x_0 > a^{1/n}$.

If we put $x_p = r + \epsilon_p$, $r = a^{1/n}$, we readily find from (4) that

$$\lim_{p=\infty} \frac{\epsilon_{p+1}}{\epsilon_p^3} = \frac{n^2 - 1}{12r^2}.$$

Thus, ϵ_{p+1} is of the order of ϵ_p^3.

Figure 1 shows that the rapid convergence of the sequence $\{x_p\}$ is due to the fact that the graph of $y = F(x)$ has a point of inflection with horizontal tangent at $x = a^{1/n}$. In the general case,

$$F(x) = x - \frac{f(x)}{f'(x) - \dfrac{f(x)}{f'(x)} \dfrac{f''(x)}{2}},$$

whose derivative is

$$F'(x) = \frac{f^2(3f''^2 - 2f'f''')}{(2f'^2 - ff'')^2}.$$

Therefore, in general, the graph of $y = F(x)$ has a point of inflection with horizontal tangent at a root of the equation $f(x) = 0$.

EXAMPLE 2. *To compute the positive real root of the reduced cubic equation* $x^3 + bx - c = 0$, b, c *real*, $b \neq 0$, $c > 0$. Here, Newton's formula is

$$x_{p+1} = \frac{2x_p^3 + c}{3x_p^2 + b}, \tag{5}$$

and formula (2) is now

$$x_{p+1} = \frac{3x_p^5 - bx_p^3 + 6cx_p^2 + bc}{6x_p^4 + 3bx_p^2 + 3cx_p + b^2}. \tag{6}$$

If $b = 2$, $c = 20$, and we take $x_0 = 2$, formula (6) gives the approximations

$$x_1 = 2.46, \qquad x_2 = 2.46954551.$$

In this case, formula (5) gives, starting with $x_0 = 2$, the values

$$x_1 = 2.6, \qquad x_2 = 2.47, \qquad x_3 = 2.469546.$$

The value of the root, to nine decimal places, is 2.469545649.

If $x_0>0$, $3x_0^2+b>0$, then the sequence $x_1, x_2, x_3, \cdots$ given recurrently by (6) *converges to the positive real root r of the equation $x^3+bx-c=0$. The sequence is increasing if $x_0<r$, and is decreasing if $x_0>r$. If ϵ_k is the error committed in taking x_k as an approximation to r, then ϵ_{p+1} is of the order of ϵ_p^3.*

On writing the equation $x^3+bx-c=0$ in the form

$$x=\frac{c}{x^2}-\frac{b}{x},$$

and using the fact that $c>0$, we conclude immediately that the equation has

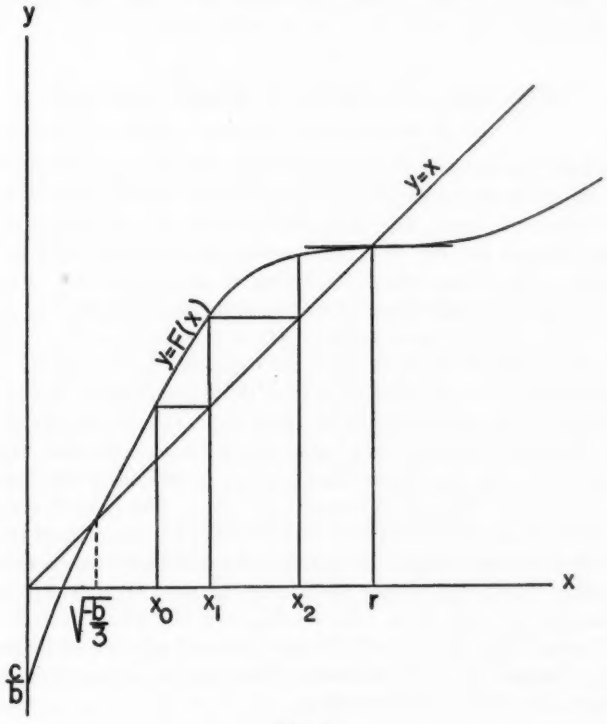

Fig. 2

just one positive real root r. This is the abscissa of the point of intersection in the first quadrant of the straight line $y=x$ and the curve $y=(c/x^2)-(b/x)$.

Consider the function

$$F(x)=\frac{3x^5-bx^3+6cx^2+bc}{6x^4+3bx^2+3cx+b^2}.$$

Since $6x^4+3bx^2+3cx+b^2=6(x^2+b/4)^2+3cx+5b^2/8>0$ for $x>0$, we see that $F(x)$ is continuous for $x>0$. Also, $F'(x)>0$ for $x>0$ except that $F'(r)=0$. It is evident from the graph that $x_p \uparrow r$ or $x_p \downarrow r$ according as $x_0<r$ or $x_0>r$, respectively. We find that $\lim_{p=\infty}[\epsilon_{p+1}/\epsilon_p^3]=(6r^2-b)/(6r^4+3br^2+3cr+b^2)$, so that ϵ_{p+1} is of the order of ϵ_p^3.

AUTHOR'S NOTE. After this article was in type, my attention was called to a note by J. S. Frame (this MONTHLY, vol. 51 (1944), No. 1, Part I, pp. 36–38) containing precisely this idea for extending Newton's formula. The above discussion brings to light other properties of Frame's formula. In addition to these, it may be easily shown that if r is a simple (real or imaginary) root of a polynomial $f(x)$, and if x_0 is taken sufficiently near to r, then the sequence (2) converges to r in such a way that $|x_{p+1}-r|$ is of the order of $|x_p-r|^3$.

EDITORIAL—THE PROOF OF EULER'S EQUATION

C. B. ALLENDOERFER, Haverford College

Judging from the volume of mail addressed to "Classroom Notes" it appears that one of the greatest mysteries of undergraduate mathematics is the equation of Euler: $e^{ix}=\cos x+i \sin x$. The objective of these correspondents is to develop this formula without the use of infinite series; and judging from the desperate devices employed by these writers in seeking to attain this end, it is highly desirable that a rigorous simple proof of this formula be available. The present note is therefore dedicated to those authors whose papers have been rejected, but whose ideas have helped me to prepare the following presentation.

The first point to be emphasized is that the expression e^{ix} has to be *defined*, and that certain properties must be ascribed to it. Otherwise any proof falls to the ground. Rigorous treatments of this appear in the classical literature; for example, see G. H. Hardy, *Pure Mathematics*, p. 409 (fifth edition), or E. T. Whittaker and G. N. Watson, *Modern Analysis*, p. 581 (fourth edition). Since these have more general objectives in view, it may be complained that they are too complicated for the purpose of defining the relatively simple expression e^{ix}. If, on the other hand, one wishes simplicity, he may straight off define e^{ix} to be the expression $\cos x+i \sin x$. This would settle the whole matter, but such a definition is unsatisfactory on intuitive grounds and appears to be drawn out of the air. It is hoped that the following definition is satisfactory on all three grounds: rigor, simplicity, and intuition.

DEFINITION. *e^{ix} is a complex valued function of the real variable x having the properties*:

$$e^{i0} = 1; \tag{1}$$

$$de^{ix}/dx = ie^{ix}. \tag{2}$$

THEOREM. $e^{ix}=\cos x+i \sin x$.

Proof: Let $e^{ix}=\rho(\cos\theta+i\sin\theta)$, $0\leqq\theta<2\pi$; $\rho\geqq 0$. From property (1) of the definition:

$$\text{when} \quad x=0; \qquad \text{then} \quad \rho=1 \quad \text{and} \quad \theta=0.$$

From property (2) of the definition:

$$\frac{d\rho}{dx}(\cos\theta+i\sin\theta)+\rho(-\sin\theta+i\cos\theta)\frac{d\theta}{dx}=i\rho(\cos\theta+i\sin\theta).$$

Equating real and imaginary parts we have:

$$\frac{d\rho}{dx}\cos\theta-\rho\sin\theta\frac{d\theta}{dx}=-\rho\sin\theta$$

$$\frac{d\rho}{dx}\sin\theta+\rho\cos\theta\frac{d\theta}{dx}=\rho\cos\theta.$$

Solving, we obtain

$$\frac{d\rho}{dx}=0; \qquad \rho\frac{d\theta}{dx}=\rho.$$

Hence: $\rho=1$ and $\theta=x$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 801. *Proposed by J. P. Ballantine, University of Washington*

How many different bidding sequences are possible in one hand of Bridge?

E 802. *Proposed by E. A. Nordhaus, Michigan State College*

(1) Find the smallest positive integer N having the property that the sum of its digits does not divide the sum of the cubes of its digits.

(2) Find the two consecutive positive integers each of which equals the sum of the cubes of its digits.

E 803. *Proposed by J. H. Butchart, Arizona State College at Flagstaff*

If the squares of the sides of a triangle form an arithmetic progression, the line joining the centroid and the symmedian point is parallel to one side of the triangle.

E 804. *Proposed by S. H. Gould, University of Wisconsin*

Denote by U the ellipsoid $a_1^2x_1^2+a_2^2x_2^2+a_3^2x_3^2=1$, by E_b the ellipse of intersection of U with the plane $b_1x_1+b_2x_2+b_3x_3=0$, by (p_1, p_2, p_3) a point variable on E_b, and by E_p the ellipse of intersection of U with the plane $p_1x_1+p_2x_2+p_3x_3=0$.

Determine (p_1, p_2, p_3) so as to minimize the major axis of E_p.

E 805. *Proposed by N. A. Court, University of Oklahoma*

If two coplanar edges of a tetrahedron are each equal to the respectively opposite edge, the remaining two opposite edges are each coplanar with the Euler line of the tetrahedron.

SOLUTIONS

Quadrangle Inscribed in an Equilateral Hyperbola

E 769 [1947, 224]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a plane quadrangle $ABCD$, the perpendicular at A to side AB cuts the opposite side CD in M, and the perpendicular at A to side AD cuts the opposite side BC in N. Show that the radical axis of the circles described on AM and AN as diameters coincides with the tangent at A to the equilateral hyperbola circumscribing the quadrangle.

Solution by L. M. Kelly, University of Missouri. Place the figure on a rectangular cartesian frame of reference so that the circumscribed equilateral hyperbola assumes the standard position $xy=k$, and let the coordinates of A, B, C, D be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively.

The circles on AM and AN as diameters intersect in a point on MN, so our problem is, evidently, to show that the tangent at A is perpendicular to MN. Draw lines parallel to the y-axis through B and D meeting AN and AM in P and P' respectively. Also draw lines through B and D parallel to the x-axis meeting AN and AM in Q and Q' respectively. The pencil $D(A\infty_y C\infty_x)$ is projectively related to the pencil $B(A\infty_y C\infty_x)$, whence the ranges $APNQ$ and $AP'MQ'$ are projectively related and, having A as a common corresponding point, are in perspective position. That is, PP', MN, and QQ' are concurrent. We propose now to show that the slopes of PP' and QQ' are both equal to x_1^2/k. This will establish the proposition.

The equation of AN is

$$\begin{aligned}(y-y_1)/(x-x_1) &= -(x_1-x_4)/(y_1-y_4)\\ &= -(x_1-x_4)/(k/x_1-k/x_4) = x_1x_4/k.\end{aligned}$$

Similarly, the equation of AM is

$$(y-y_1)/(x-x_1) = x_1x_2/k.$$

The coordinates P, P', Q, Q' are then found to be

$$P: \quad [x_2,\ x_1x_4(x_2-x_1)/k+y_1],$$

$$P': \quad [x_4,\ x_1x_2(x_4-x_1)/k+y_1],$$

$$Q: \quad [k(y_2 - y_1)/x_1x_4 + x_1,\ y_2],$$
$$Q': \quad [k(y_4 - y_1)/x_1x_2 + x_1,\ y_4].$$

The solution is now easily completed.

The proposer stated that this problem is a particular case suggested by a more general theorem given by M. A. de Majo, *L'Intermédiare des recherches mathématiques*, Paris, 1946, p. 45.

Derivatives of a Function

E 770 [1947, 224]. *Proposed by Norman Miller, Queen's University, Canada*

A function $f(x)$ has the following definition, n being a positive integer,

$$f(x) = x^n \sin \pi/x, \qquad x \neq 0,$$
$$f(0) = 0.$$

Show that, if $n=2m$ or $2m+1$ (m a positive integer), then at $x=0$ the function possesses a derivative of the mth but no higher order, and that, at $x=0$, $f^{(m)}(x)$ is continuous or discontinuous according as $n=2m+1$ or $n=2m$.

Solution by S. T. Thompson, Tacoma, Washington. We may show, by mathematical induction, that, for $x \neq 0$ and $0 \leqq r \leqq m$,

$$f^{(r)}(x) = P_r \sin \pi/x + Q_r \cos \pi/x,$$

where P_r and Q_r are polynomials in x, in which the term of lowest degree in either polynomial is of degree $n-2r$. Again, by mathematical induction, we may easily show that

$$f^{(r)}(0) = 0, \qquad 0 \leqq r \leqq m.$$

(a) Suppose $n=2m$. Then, if we take $r=m$, either P_m or Q_m contains a term of degree zero, whence $\lim_{x\to 0} f^{(m)}(x)$ does not exist. Thus $f^{(m)}(x)$ is discontinuous at the origin, and, of course, $f^{(m+1)}(x)$ then cannot exist at the origin.

(b) Suppose $n=2m+1$. Then, if we take $r=m$, all terms of P_m and Q_m are of degree greater than zero. Hence $\lim_{x\to 0} f^{(m)}(x) = 0$, and $f^{(m)}(x)$ is continuous at the origin. However,

$$f^{(m+1)}(0) = \lim_{x\to 0} \frac{f^{(m)}(x) - f^{(m)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(m)}(x)}{x}$$
$$= \lim_{x\to 0} [(P_m/x) \sin \pi/x + (Q_m/x) \cos \pi/x].$$

Since P_m/x or Q_m/x contains a term of degree zero, this limit clearly does not exist, and therefore $f^{(m+1)}(x)$ does not exist at the origin.

Also solved by Elizabeth M. Huskey, E. S. Keeping, and the proposer.

Keeping pointed out that this problem, for $n=2$, is discussed in vol. 1 of de La Vallée Poussin's *Cours d'Analyse*.

The Most Likely Total

E 771 [1947, 280]. *Proposed by C. C. Carter, Bluffs, Illinois*

A die bearing the numbers 0, 1, 2, 3, 4, 5 on its faces is repeatedly thrown until the total of the throws first exceeds 12. What is the most likely total that will be thus obtained?

Solution by N. J. Fine, University of Pennsylvania. Consider the throw before the last; the total on that throw must be 12, 11, 10, 9, or 8. If the total is 12, the final result must be 13, 14, 15, 16, or 17, with an equal chance for each. Similarly, if the total is 11, the final result must be 13, 14, 15, or 16, with an equal chance for each, and so on. It is now clear that the most likely final result is 13.

Also solved by E. M. Berry, Monte Dernham, Abraham Golub, J. B. Kelly, J. M. Kingston, William Kruskal, and C. R. Phelps.

In addition to solving the given problem, Dernham considered the problem of computing the probabilities of obtaining the totals 13, 14, 15, 16, 17. These probabilities he found to be fractions with denominator 1,220,703,125 $(=5^{13})$ and respective numerators 408,335,461; 327,125,461; 243,735,961; 160,543,561; 80,962,681. He arrived at these results by considering the essentially equivalent problem: *From a bag containing* 5 *marbles, numbered respectively* 1, 2, 3, 4, 5, *a marble is drawn at random and replaced. This is repeated until the total of the numbers drawn exceeds* 12. *What are the respective probabilities of obtaining the totals* 13, 14, 15, 16, 17?

A number of solvers generalized the given problem by replacing 12 by N. The answer is then $N+1$.

Shortest Paths in a Lattice

E 772 [1947, 281]. *Proposed by D. H. Browne, Buffalo, N.Y.*

What is the number of shortest paths between two points in an n-dimensional lattice?

I. *Solution by Norman Miller, Queen's University.* Let each cell in the lattice have an edge of one unit. A shortest path between two points is made up of a number a_1 of units in a direction d_1, a number a_2 in a direction d_2, $\cdots$, a number a_n in a direction d_n. If we take a_1 identical letters d_1, a_2 identical letters d_2, $\cdots$, a_n identical letters d_n, then any distinct arrangement, in a line, of the $a_1+a_2+\cdots+a_n$ letters will represent a distinct one of the required paths. The number is therefore

$$(a_1 + a_2 + \cdots + a_n)!/(a_1!a_2! \cdots a_n!).$$

II. *Solution by J. B. Kelly, Kew Gardens, N. Y.* Let $R(a_1, a_2, \cdots, a_n)$ be the number of shortest paths between $(0, 0, \cdots, 0)$ and $(a_1, a_2, \cdots, a_n)$, where $a_1, a_2, \cdots, a_n$ are integers. There is no loss in generality in assuming that $a_1, a_2, \cdots, a_n$ are non-negative. Any shortest path between the two points must enter $(a_1, a_2, \cdots, a_n)$ from one of the points $(a_1-1, a_2, \cdots, a_n)$, $(a_1, a_2-1, \cdots, a_n)$, $\cdots$, $(a_1, a_2, \cdots, a_n-1)$. Hence

(1) $$R(a_1, \cdots, a_n) = R(a_1 - 1, a_2, \cdots, a_n) + \cdots + R(a_1, a_2, \cdots, a_n - 1).$$

Obviously

(2) $$R(a_1, 0, 0, \cdots, 0) = R(0, a_2, 0, \cdots, 0) = \cdots = R(0, 0, 0, \cdots, a_n) = 1.$$

The unique solution of the difference equation (1) with the boundary conditions (2) is

$$R(a_1, \cdots, a_n) = (a_1 + a_2 + \cdots + a_n)!/(a_1!a_2! \cdots a_n!).$$

Also solved by Harry Goheen, R. B. Herrera, Leo Moser, C. R. Perisho, and the proposer.

Noughts and Crosses

E 773 [1947, 281]. *Proposed by A. L. Rubinoff, University of Toronto*

Suppose that noughts and crosses are played on an n-dimensional cube of side k. Show that there are precisely

$$\frac{(k+2)^n - k^n}{2}$$

rows, columns, diagonals, . . . on which a win may be scored.

Solution by Leo Moser, University of Manitoba. Consider the "cube" of side k inside a cube of side $k+2$. Clearly, every win will determine exactly one pair of surface elements, while each surface element determines exactly one win. Hence the number of wins will be half the number of surface elements, which is the result stated.

Also solved by N. J. Fine and the proposer.

A Positive Determinant

E 775 [1947, 281]. *Proposed by R. P. Boas, Jr., Brown University*

Consider a determinant of order n whose elements are x on the main diagonal, ± 1 elsewhere. Find the smallest positive number a such that for $x > a$ the determinant is positive for all choices of the $\pm$ signs.

Solution by Olga Taussky (Mrs. John Todd), London, England. It is known that the determinant $d(a_{ij})$ is positive if $a_{ii} > \sum_{i \neq j} |a_{ij}|$, for $i = 1, 2, \cdots, n$. It follows that for $x > n-1$ the determinant in question is certainly positive for all choices of signs. We can take $a = n-1$ because the determinant clearly vanishes if $x = n-1$ and all the signs are negative.

The result quoted is due to P. Furtwängler (*Sitzunsber. Acad. Wiss. Wien*, Abt. II *a*, 145 (1936), p. 527) and can be proved by induction.

Related results have been discovered by Minkowski, Hadamard, Artin, and others. There recently were some notes about these in the *Mathematical Gazette* (XXVI, p. 191; XXVIII, p. 63; XXIX, p. 15).

Also solved (partially) by R. P. Brady and Norman Miller.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4280. *Proposed by Joseph Rosenbaum, Weatogue, Connecticut*

The points $A_1, A_2, \cdots, A_n$ are pursuing one another cyclically, the speed of A_i being proportional to the distance A_iA_{i+1}. If at time $t=0$ the points are the vertices of a given polygon, find the paths described by the points.

4281. *Proposed by M. S. Knebelman, Washington State College*

Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

4282. *Proposed by F. W. Herlihy, Comstock, Michigan*

An elastic string (modulus λ, mass ma, unstretched length a) is confined within a straight tube to one end of which it is fastened. The tube rotates around that end with uniform angular velocity ω in a horizontal plane. Find the length of the string in equilibrium.

4283. *Proposed by E. P. Starke, Rutgers University*

The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of C, $f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

4284. *Proposed by E. T. Bell, California Institute of Technology*

"Diophantus [II, 20 and IV, 45] proposed to find three squares such that

$$y^2 - x^2 : z^2 - y^2 = a : b,$$

where $a:b$ is a given ratio." (L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 419.) For each of $a/b=3, 1/3$ he gave one rational solution. Prove that the complete integer solution is given by

$$\begin{aligned} 2x &= k(a_1b_1g^2 + 2agh - a_2b_2h^2), \\ 2y &= k(a_1b_1g^2 + a_2b_2h^2), \\ 2z &= k(a_1b_1g^2 - 2bgh - a_2b_2h^2), \end{aligned}$$

where a_1, a_2 are any integers whose product is a, and b_1, b_2 are any integers whose

product is b, and k (for any pair of integer values of g, h) is an arbitrary integer multiple of the reciprocal of the G. C. D. of b_1g+a_2h, a_1g-b_2h, $a+b$.

SOLUTIONS

Special Tetrahedrons

4013 [1941, 639]. *Proposed by V. Thébault, Tennie, Sarthe, France*

Determine the kind of tetrahedron for which: (1) The circumcenter (or incenter) is on one of the medians. (2) The straight line joining the circumcenter to the centroid is perpendicular (or parallel) to one of the faces.

*Solution by R. Bouvaist, Vincelles (Saône-et-Loire), France.** Let $ABCD$ be the tetrahedron, O its circumcenter, G its centroid.

1. If O is on the median AG, the polar plane of G with respect to the circumsphere (O) is parallel to the tangent plane of (O) at A. The equation of (O) in tetrahedral normal coördinates (with $ABCD$ chosen as the reference tetrahedron) is†

$$\frac{a^2yz}{h_2h_3}+\frac{b^2zx}{h_3h_1}+\frac{c^2xy}{h_1h_2}+\frac{a'^2xt}{h_1h_4}+\frac{b'^2yt}{h_2h_4}+\frac{c'^2zt}{h_3h_4}=0,$$

where $BC=a$, $AD=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$, and h_1, h_2, h_3, h_4 designate the altitudes drawn from A, B, C, D. The equations of the polar plane of G with respect to (O) and of the tangent plane of (O) at A are, respectively,

$$L_1\equiv\sum(a'^2+b^2+c^2)x/h_1=0,$$

$$L_2\equiv\frac{c^2}{h_2}y+\frac{b^2}{h_3}z+\frac{a'^2}{h_4}t=0.$$

These planes are parallel if

$$\frac{a^2-a'^2}{b^2+c^2-a'^2}=\frac{b^2-b'^2}{b^2-c^2-a'^2}=\frac{c^2-c'^2}{c^2-a'^2-b^2}\cdot\ddagger$$

The incenter I will be on the median AG if $h_2=h_3=h_4$.

2. The line OG will be perpendicular to BCD if the polar plane of G with respect to (O) is parallel to BCD, that is, if

$$-(a^2-a'^2)=b^2-b'^2=c^2-c'^2.$$

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* Translated and checked by W. E. Byrne, Virginia Military Institute.

† See Niewenglowski, *Cours de Géométrie Analytique*, t. III, Art. 120.

‡ The condition that L_1 and L_2 are parallel is that L_1, L_2, and L_3 be linearly dependent, where L_3 is the plane at infinity whose equation is

$$L_3\equiv\frac{x}{h_1}+\frac{y}{h_2}+\frac{z}{h_3}+\frac{t}{h_4}=0.$$

We have

$$L_1-(a'^2+b^2+c^2)L_3\equiv 2\lambda L_4,$$

from which the above condition is easily obtained.

OG will be parallel to BCD if the polar plane of G with respect to (O) is parallel to the altitude drawn from A, that is, if

$$\frac{a'^2+b^2+c^2}{h_1}=\frac{a^2+b'^2+c^2}{h_2}\cos c'+\frac{a^2+b^2+c'^2}{h_3}\cos b'+\frac{a'^2+b'^2+c'^2}{h_4}\cos a,$$

where c', b', a are the plane angles of the dihedral angles of the edges DC, DB, BC.

Circles Related to the Triangle

4092 [1943, 457]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For the triangle ABC let $(A_1B_1C_1)$, $(A_2B_2C_2)$, $\cdots$, $(A_nB_nC_n)$ be the centers of squares constructed exteriorly (or interiorly) on the sides (BC, CA, AB), (B_1C_1, C_1A_1, A_1B_1), $\cdots$, (B_nC_n, C_nA_n, A_nB_n) of the corresponding triangles. (1) Show that the center ω_1 of the circle orthogonal to the circles with centers A, B, C and radii B_1C_1, C_1A_1, A_1B_1 coincides with the center of the nine-point circle of ABC. (2) Find the locus of centers $\omega_2, \omega_3, \cdots, \omega_n$ of the circles, orthogonal to the circles with centers, A, B, C and with radii (B_2C_2, C_2A_2, A_2B_2), (B_3C_3, C_3A_3, A_3B_3), $\cdots$, (B_nC_n, C_nA_n, A_nB_n).

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Let M_1, M_2, M_3 be the midpoints of BC, CA, AB; H_1, H_2, H_3 the feet of the altitudes upon the same sides. We consider the squares drawn exteriorly, and suppose $a>b>c$. Then

$$\overline{B_1C_1}^2=\overline{AB_1}^2+\overline{AC_1}^2-2AB_1\cdot AC_1\cos B_1AC_1,$$

$$\overline{B_1C_1}^2=(b^2+c^2)/2+bc\sin A=(b^2+c^2+4S)/2,$$

where S is the area of ABC. Similarly

$$\overline{C_1A_1}^2=(c^2+a^2+4S)/2, \qquad \overline{A_1B_1}^2=(a^2+b^2+4S)/2.$$

The radical axis of the circles (B, C_1A_1), (C, A_1B_1) cuts BC in X_1. Then

$$\overline{A_1B_1}^2-\overline{A_1C_1}^2=(b^2-c^2)/2=M_1H_1\cdot BC=2a\cdot M_1X_1.$$

Hence $2M_1X_1=M_1H_1$, and similarly $2M_2X_2=M_2H_2$, $2M_3X_3=M_3H_3$. The radical center is therefore the center ω of the nine-point circle. Again

$$\overline{A_2B_2}^2-\overline{A_2C_2}^2=(\overline{A_1B_1}^2-\overline{A_1C_1}^2)/2=(b^2-c^2)/4.$$

If the radical axis of the circles (B, C_2A_2), (C, A_2B_2) cuts BC in X_1',

$$4M_1X_1'=M_1H_1, \qquad 4M_2X_2'=M_2H_2, \qquad 4M_3X_3'=M_3H_3.$$

Hence in general

$$2^nM_1X_1^{(n)}=M_1H_1, \qquad 2^nM_2X_2^{(n)}=M_2H_2, \qquad 2^nM_3X_3^{(n)}=M_3H_3.$$

* Translated and checked by C. J. Ramler, Catholic University of America, Washington, D. C.

The locus of the radical centers is then the segment ω_0 joining the center of the nine-point circle to the circumcenter of ABC.

Euler's Function

4221 [1946, 537]. *Proposed by P. Erdös, Syracuse University.*

Let $\phi(n)=n\prod_{p|n}(1-1/p)$ be Euler's function. Prove that for every k the equation $\phi(x)=k!$ is solvable.

Solution by J. Lambek, McGill University, Montreal. Consider the problem of solving the equation

$$\phi(x) = n \tag{1}$$

subject to the condition that x has precisely the same prime factors as n. The restriction on x implies that

$$\phi(x)/x = \phi(n)/n. \tag{2}$$

By (1) and (2) it follows that

$$x = n^2/\phi(n). \tag{3}$$

Let $n=\prod p^{\alpha}$. Then $x=\prod p^{\alpha+1}/(p-1)$. The integer x will have the same prime factors as n if and only if

$$\prod_{p|n} (p-1) \mid n. \tag{4}$$

Conversely (3) and (4) imply (1) and (2). Therefore a necessary and sufficient condition for (1) to be solvable under the given restriction is (4), the unique solution being given by (3).

If $n=k!$, (4) is clearly satisfied. An explicit solution of the proposed problem is therefore given by

$$x = (k!)^2/\phi(k!).$$

Solved also by P. T. Bateman, Alfred Brauer, N. J. Fine, William Gustin, R. B. Herrera, V. L. Klee, Jr., Emma Lehmer, Leo Moser, Ivan Niven, and W. V. Parker.

Volume of a Simplex in Terms of Its Edges

4222 [1946, 537]. *Proposed by J. H. Butchart, Grinnell College*

If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$

respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

Solution by Leo Zippin and Joan Williamson, Queens College. We shall prove the following: *Let V_n be the volume of a simplex with vertices $P_1, P_2, \cdots, P_{n+1}$ in Euclidean space of n dimensions. If*

$$A = (a_{ij}), \qquad i, j = 1, 2, \cdots, n+1,$$

is the square matrix of order $n+1$ with $a_{ij}=\overline{P_iP_j}^2$ and if ϵ is the row vector $(1, 1, \cdots, 1)$ of dimension $n+1$ and ϵ' is the column vector which is the transpose of ϵ, then

$$\begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} = (-1)^{n+1}2^n(n!)^2V_n^2.$$

In a rectangular coördinate system let the coördinates of P_i be given by the row vector $x_i=(x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)})$ and let C be the matrix of n columns and $n+1$ rows whose rows are the $n+1$ vectors x_i. Then the ordinary determinantal expression for V_n takes the form

$$n!V_n = |C\epsilon'|. \tag{1}$$

Further

$$CC' = B = (b_{ij}), \qquad i, j = 1, 2, \cdots, n+1, \tag{2}$$

where

$$b_{ij} = \sum_{k=1}^{n} x_i^{(k)}x_j^{(k)}.$$

If, O being the origin, we put $OP_i^2=r_i^2$ and $\rho=(r_1^2, r_2^2, \cdots, r_{n+1}^2)$, we have

$$r_ir_j\cos P_iOP_j = \sum_{k=1}^{n} x_i^{(k)}x_j^{(k)} = b_{ij}, \qquad i, j = 1, 2, \cdots, n+1, \tag{3}$$

and

$$D = (d_{ij}) = \rho'\epsilon + \epsilon'\rho, \tag{4}$$

where $d_{ij}=r_i^2+r_j^2$. Since

$$a_{ij} = \overline{P_iP_j}^2 = r_i^2 + r_j^2 - 2r_ir_j\cos P_iOP_j,$$

we have by (3) and (4)

$$A = D - 2B. \tag{5}$$

If E is the unit matrix of order $n+1$, we have

$$\begin{pmatrix} E & -\rho' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & \epsilon' \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} E & 0 \\ -\rho & 1 \end{pmatrix} = \begin{pmatrix} A - \rho'\epsilon - \epsilon'\rho & \epsilon' \\ \epsilon & 0 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}$$

by (5). Therefore

$$\begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} = \begin{vmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{vmatrix}. \tag{6}$$

Finally,

$$\begin{pmatrix} -2C & 0 & \epsilon' \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} C' & 0 \\ \epsilon & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2CC' & \epsilon' \\ \epsilon & 0 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}$$

by (2). Therefore by (6)

$$\begin{aligned} \begin{vmatrix} A & \epsilon' \\ \epsilon & 0 \end{vmatrix} &= -\begin{vmatrix} -2C & \epsilon' \end{vmatrix}\begin{vmatrix} C' \\ \epsilon \end{vmatrix}, \\ &= (-1)^{n+1}2^n\begin{vmatrix} C & \epsilon' \end{vmatrix}^2 \\ &= (-1)^{n+1}2^n(n!)^2V_n^2 \end{aligned}$$

by (1)

It is interesting to note that the proof can be considerably simplified if the assumption is made that $V_n \neq 0$. Then the origin may be taken as the center of the circumscribing hyper-sphere, whence $r_i^2 = r^2$, $i = 1, 2, \cdots, n+1$. Then (6) is obtained from the matrix equation

$$\begin{pmatrix} A & \epsilon' \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} E & 0 \\ -2r^2\epsilon & 1 \end{pmatrix} = \begin{pmatrix} -2B & \epsilon' \\ \epsilon & 0 \end{pmatrix}.$$

Editorial Note. W. V. Parker calls attention to the proof for the case $n = 3$ given in Kowalewski, *Einführung in die Determinantentheorie*, pp. 344–350.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, 27, N. Y. and not to any of the other editors or officers of the Association.

Differential and Integral Calculus. By F. D. Murnaghan. Brooklyn, The Remsen Press, 1947. 10+502 pages. $5.00.

In this book one finds an admirable, courageous, and, in the opinion of the reviewer, a completely successful attempt to present the calculus correctly, and at the same time, in a form intelligible to the beginning student.

The first departure from the methods to be found in the currently popular texts is the lucid explanation of the concept of a real number. Starting with the implicit and justifiable assumption that the student's intuitive notions concern-

ing counting numbers, integers, and rational numbers are correct, the author gives a brief (13 pages including 46 exercises) discussion of these concepts which states and explains the salient facts concerning them, as well as demonstrates the inadequacy of the rational number system for certain types of problems. In this discussion the student is asked to learn to distinguish between the positive integer 5 and the counting number 5, but the text is directed to the student and he is not subjected to the hair-splitting subtleties which often puzzle beginning graduate students. The exercises here are well graded in difficulty ranging from the first "Can the calculation indicated by the equation $x+7=11$ be performed in counting numbers?" to the last "Show that if the product ab of two positive integers is divisible by a prime number p then either a or b (possibly both) is divisible by p." Here and throughout the text hints are given when necessary and, in the case of exceptionally difficult problems, a complete solution is outlined.

The next twenty pages are devoted to the concept of a real number which is explained by means of nested sequences of rational intervals. Addition and multiplication of real numbers are defined and it is proved that $x(y+z)=xy+xz$, but the text is not encumbered with the tedious verification of all of the algebraic relations which define a field. Instead, the most important of these laws are stated as exercises. The student at this point cannot be expected to have much appreciation of the power of the concept of a real number, nor will he have mastered the technique of using this concept. However, after having used it many times at judiciously spaced intervals throughout the text even the mediocre student should appreciate its value. This technique appears for example in the proofs of the fundamental properties of continuous functions, in the proof of the integrability of continuous functions and in showing that local boundedness on a finite closed interval implies boundedness over the interval.

Another innovation consists of treating the concepts of differentiability, differential, and derivative, as well as formal drill on differentiation, before introducing the concepts of limit and continuous function. This is done by using the notion of null function (one whose limit is zero) as the basic idea.

Beside the topics usually found in a first course in the calculus which is designed to go through the differentiation and integration of the trigonometric, inverse trigonometric, logarithmic and exponential functions, the text contains a chapter on Special Plane Curves and one on Infinite Series and Integrals.

To the reviewer, who is accustomed to teaching "theory" to freshmen, it appears that there is no place in the text where the student gets an overdose of the abstract and more difficult parts of the calculus. The purely manipulative parts are well interspersed and in all there are over twelve hundred problems. It is apparent that the presentation of the calculus as found in this text has been well tested by classroom use and that Professor Murnaghan has spent much time and thought on the many problems involved in teaching undergraduate mathematics.

NELSON DUNFORD

Curso de Análisis Matemático. Tomo II. Por Cristóbal de Losada y Puga. Lima, Universidad Católica del Peru, 1947. 21+701 pages. $9.50.

The first volume was reviewed in this MONTHLY, vol. 53 (1946), pp. 268–279. The second volume continues in the same careful and clear style as its predecessor; the printing and illustrations are of the same high quality.

The volume is divided into three main parts: *Series*, *Geometric Applications of the Differential Calculus*, *Questions Concerning Integration*. The material is in general such as is customary in a course in advanced calculus.

The first part begins with standard material on convergence of series, with a chapter on uniform convergence; power series; double series; infinite products. The next two chapters are principally devoted to Taylor series in one and several variables, with applications. Weierstrass's nondifferentiable function is discussed in detail.

The second part discusses first plane curves, then space curves, and finally surfaces. The exposition is given in logical order rather than the usual textbook order, so that topics which are often treated separately as calculus and as differential geometry appear together.

The third part opens with a long chapter on numerical, graphical and mechanical integration. The error involved in Simpson's rule is discussed. Several types of planimeters are described and their theory is given. A chapter on improper integrals leads the author to give a proof of the transcendence of e. Chapters on the manipulation of integrals containing a parameter and on line integrals follow. The book ends with a chapter on multiple integrals and their transformation. The author gives Schwarz's example to show that the area of a surface cannot be defined as the limit of the areas of inscribed polyhedra.

R. P. BOAS, JR.

College Algebra. By M. Richardson. New York, Prentice-Hall, Inc., 1947. 16+472 pages. $2.85.

This book appears attractive, not too bulky, and clearly readable, as far as the printing and binding are concerned. It pioneers in a new road of explanation and arrangement of material, which may be a pattern for some courses in college algebra in the not too distant future.

An early definition of the usual mathematical symbols and Greek alphabet in the *Preface for the Student* is most timely and appreciated. An unusual effort in *The Number System of Algebra* (Chapter I) is made to acquaint the student with many of our usually accepted procedures and rules of high school algebra. To understand and appreciate these will strain the average freshman's intellect a trifle! Those who succeed will be well repaid for their effort. Such a fact as the irrationality of the square root of two comes in for several interesting pages of comment in this chapter. On page 27 the motivation for the definition of the product of two negative numbers is welcome for better college students. Algebra is treated as *a language, a logical science, a collection of techniques of calculation, a*

branch of human endeavor, as well as a collection of puzzles in Chapter II. It makes good reading material.

Exponents are appropriately explained in Chapter III. Wise emphasis on equations and identities (page 66), generous explanations of coefficient, degree of a monomial, and so on, are appreciated by the modern student. A thorough introduction to *Functions and Graphs* (Chapter IV) is followed by a discussion of the division algorithm (Chapter V) which makes factoring, as well as the least common multiple and highest common factor, in Chapter VI take on more meaning.

The analytic geometry of the linear function with slope and two point ideas attached, make Chapter VII read like part of a modern text-book on analytic geometry. Verbal problems get an extra dose of explanation before they are reached in the exercises. This has considerable merit in this decade of student weaknesses with such. In Chapter IX, the negative, zero, and fractional exponents come in for ample discussion before radicals are studied in the next chapter. *Quadratic Equations and Quadratic Functions* (Chapter XI) are handled thoroughly and *Systems of Equations in Two Unknowns* follow in the next chapter.

The usual treatment of *Ratio, Proportion, and Variation* in Chapter XIII and of *Complex Numbers* in Chapter XIV are given. In Chapter XV, a nice illustration of finding an irrational root of a polynomial equated to zero by use of synthetic division is given on pages 245 and 246. This required some tedious computation but students will appreciate its meaning. Horner's method was made easier by an explanation of linear interpolation.

Determinants and Elimination Theory of Chapter XVI have an unusually thorough treatment for a book of this nature, while *Permutations and Combinations* (Chapter XVII) and *Probability* (Chapter XVIII) remind one of the college algebra text-books of two decades ago. (The reviewer does not object to this, however.)

The soldier illustration of Chapter XIX makes mathematical induction take on real meaning for every reader. Much can be said in favor of discussing both this topic and the binomial theorem at this stage of the student's mathematical knowledge. Next follow *Progressions* (Chapter XXI), *Inequalities* (Chapter XXII), *Logarithms* (Chapter XXIII), and *Mathematics of Investment* (Chapter XXIV) with adequate treatments. Few college algebra books give a chapter to the *Euclidean Algorithm* (Chapter XXV), while *Partial Fractions* (Chapter XXVI) and *Series* (Chapter XXVII) are not uncommon topics for some of the older books as well as the newer better books.

Interpolation and Curve Fitting (Chapter XXVIII) includes the unusual treatment of the Lagrange interpolation formula, as well as other more common topics, apparently well discussed.

One is overwhelmed with a book of this type if he hopes an ordinary freshman will digest this material in a usual one semester course. It would seem more likely that this material would keep one busy for two semesters and then

he and the instructor would be grateful if an acceptable digestion process is realized! On the other hand, those topics which normally would be omitted in a one semester course will serve as valuable reference material for the student who continues with the study of mathematics.

W. R. HUTCHERSON

Calculating Machines. By D. R. Hartree. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 40 pages. $.75.

This publication of the inaugural lecture of D. R. Hartree as Plummer Professor of Mathematical Physics in the University of Cambridge carries the subtitle, *Recent and Prospective Developments and their impact on Mathematical Physics*. In 35 small pages of text and two pictures (of the ENIAC), Professor Hartree provides a brief but comprehensive introduction to some of the problems associated with large scale computing machines.

After a comparison of the relative advantages of analogue and digital machines, the types of operations which components of a digital machine must perform are enumerated and discussed. The ENIAC provides examples of parts which perform these operations. Inadequacies of present machines and the consequent developments are mentioned. A set of simultaneous non-linear differential equations with two point boundary conditions on which Professor Hartree and the ENIAC worked during the summer of 1946 is used to illustrate the power of high speed calculating machines. This problem and the problem of solving simultaneous linear algebraic equations are discussed later in connection with the concept of the simplicity of any particular formulation of a mathematical problem; the simplicity of a formulation depends, of course, on the available aids to solution. The possibilities of important new lines of research in mathematics stimulated by shifts in the criteria of simplicity are emphasized. Mathematical physics will need to adjust to the new criteria of simplicity as well as to the possibility of dealing with problems heretofore involving prohibitive amounts of calculation.

K. J. ARNOLD

NEW BOOKS RECEIVED

Brief College Algebra. Revised Edition .By W. L. Hart. Boston, D. C. Heath and Co., 1947. 7+332 pages. $2.75.

Handbook of Engineering. Fourth Edition. By L. A. Waterbury. New York, John Wiley and Sons, Inc., 1947. 18+386 pages. $2.50.

Elementary Differential Equations. Third Edition. By L. M. Kells. New York, McGraw-Hill Book Co., 1947. 14+311 pages. $3.00.

An Introduction to Analytical Geometry. Vol. II. By A. Robson. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 8+215 pages. $2.50.

College Algebra. By M. Richardson. New York, Prentice-Hall, Inc., 1947. 16+472 pages. $2.85.

Elementary Concepts of Mathematics. By B. W. Jones. New York, The Macmillan Company, 1947. 13+294 pages. $4.00.

Functions of a Complex Variable. Third Edition. By T. M. Macrobert. London, The Macmillan Company, 1947. 15+390 pages. $4.50.

Theory of Functions. Revised Edition. By J. F. Ritt. New York, Kings Crown Press, 1947. 10+181 pages. $3.00.

Lehrbuch der Darstellenden Geometrie. By E. Stiefel. Basel, Birkhauser, 1947. 173 pages. 28.50 sw.fr.

Vector and Tensor Analysis. By Louis Brand. New York, John Wiley and Sons, Inc., 1947. 9+439 pages. $5.50.

A Chapter in the Theory of Numbers. By L. J. Mordell. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 31 pages. $0.40.

Fundamentals of Statistics. By T. L. Kelley. Cambridge, Harvard University Press, 1947. 16+755 pages. $10.00.

Elements of Mathematical Statistics. By C. V. L. Charlier. Translated and published by J. A. Greenwood (25 Winthrop St., Brooklyn 25, New York), 1947. 120 pages. $3.00.

Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments. Second Edition. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 44+403 pages. $7.50.

Manual of Mathematics and Mechanics. Second Edition. By G. R. Clements and L. T. Wilson. New York, McGraw-Hill Book Co., 1947. 9+349 pages. $3.25.

The Early Work of Willard Gibbs in Applied Mechanics. By L. P. Wheeler, E. O. Waters, and S. W. Dudley. New York, Henry Schuman, Inc., 1947. 7+78 pages. $3.00.

Elementary Nuclear Theory. By H. A. Bethe. New York, John Wiley and Sons, Inc., 1947. 6+147 pages. $2.50.

The Naming of the Telescope. By E. Rosen. New York, Henry Schuman, Inc., 1947. 16+110 pages. $2.50.

Yale Science, the First Hundred Years, 1701–1801. By L. W. McKeehan. New York, Henry Schuman, Inc., 1947. 8+82 pages. $2.50.

An Introduction to Definitive Philosophy and Basic Psychology. By Theodore Van Schelven. Kosmos, Amsterdam, 1947. 112 pages.

Teorica della Sopravvivensa. By F. A. Insolera. Turin, Giappichelli, Editore (Via Vasco 2 ang. Via PO 21, Torino, Italy), 1947. 7+242 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE. A request has come to the Editor of this section for a syllabus or bibliography on the construction of models and other visual aids in mathematics. In the belief that some mathematics clubs may be interested in such a syllabus, the Editor is asking those who know of sources of information on this material to kindly forward the references to him. A list will be compiled for dissemination to club members upon request.

CLUB REPORTS, 1946–1947

Pi Mu Epsilon, Montana State University

The fraternity made awards to the following outstanding freshman: Virgil Naumann, Barbara Dockery and Jean Popham. The mathematics award was won by Thomas Joyce and the physics award by Leonard Lust.

New members are Thomas Joyce and Robert Willson.

Officer: Director, David Bostwick.

Mathematics Club, New York University

In addition to several business meetings, numerous evening socials were conducted consisting of highly informative lectures by authorities in various fields of mathematical endeavor with accompanying discussion and refreshments. Among the lecturers were:

The theory of numbers, by Professor Bernard Friedman

Nim, by Professor Ralph Phillips

Some aspects of the concept of necessary and sufficient conditions, by Professor Louis Baron.

Some paradoxes and problems, by Professor Morris Kline.

Our annual Freshman Mathematics Contest, sponsored and completely administered by student Club members, was held in December. Cash prizes and awards of medals were given to the first place winner, Edmund Rosen, and to Edward Swell and Harvey Strizak, who were tied for second place.

Math X, the official spring publication, was a great success. Articles were contributed by the undergraduate student body, dealing with mathematical topics of interest both to the casual reader and the more exacting technical specialist.

Student tutoring was also conducted every afternoon from two to five o'clock by our members as a service to pupils in need of such help. This has proved a boon to students and faculty alike.

The main social event of the year, the annual boat ride to Bear Mountain, was held in June.

The officers for 1946–47 were: President, Bernice Goldberg; Vice-President, Selma Ring; Secretary, Claire Weber; Treasurer, Elizabeth Markowitz.

The officers elected for 1947–48 are: President, Sid Diamond; Vice-President, Eric Libori; Secretary, George Rubin; Treasurer, Rhoda Buser; Executive council members, Bernice Goldberg, Joseph Alper and Ruth Vure.

Pi Mu Epsilon, Northwestern University

The *Illinois Beta* Chapter of *Pi Mu Epsilon* held six regular meetings during the academic year 1946–47. Programs were as follows:

Mathematics as a language, by Professor H. T. Davis

Generalizations of $(-1)^n$, by V. C. Harris

The solar system, by Robert Howerton

Some theorems on projective geometry, by Miss D. M. De Witt

Dynamics of our solar system, by Professor Oliver J. Lee.

Francis Galton: pioneer statistician, by Mr. D. H. Leavens of the Cowles Commission for Research in Economics was presented at the initiation banquet in February at which time twenty-eight new members were initiated.

The annual undergraduate mathematical competition was held in May for which prizes consisting of books were awarded to Donald MacMillan, James Murrin, and Ernest Parker. The program at the last meeting of the year was presented by the winners of the contest and consisted of a discussion of the problems on this examination.

In addition to the regular meetings, *Pi Mu Epsilon* sponsored a "get acquainted" party in November for all mathematics students. Square dancing was the featured attraction of the evening. A joint picnic with the Mathematics Department was held in June.

Officers for 1946–47 were: President, Helen Gray; Vice-president, Roseann Grundman; Recording Secretary, Marjorie Smith; Corresponding Secretary, James Murrin; Treasurer, George Knapp. Officers for 1947–48 are: President, James Murrin; Vice-president, Donald MacMillan; Secretary, Roseann Grundman; Treasurer, Ernest Parker.

Kappa Mu Epsilon, Washburn Municipal University

Monthly meetings were held during the year 1946–47 by the *Washburn Mathematics Society*, which became the *Kansas Delta* Chapter of *Kappa Mu Epsilon* on March 29, 1947. The following programs were given:

Short cuts in simple mathematics, by Thomas Pirotte, John W. Nipps, Jr., and Harold Snider.

The binary system, by Ernest Dillman

Chinese rings and other puzzles, by Margaret E. Martinson

Russian multiplication system, by Mary Fix

Folding conics, by Frances Breneman

Tricks with a slide rule, by Dr. Paul Eberhart

Probability, by Beverly Brown

Mobius rings, by Harold Snider

Vectors, by William F. Seigle.

The history of mathematics in America by Dr. Ralph G. Sanger, Kansas

State College, was the title of an address given in March to a joint meeting of the *Society* with a group of mathematics teachers from Topeka and surrounding district schools.

The installation of the Society as the *Kansas Delta* Chapter of *Kappa Mu Epsilon* was conducted by Sister Helen Sullivan, O. S. B., National Historian of *Kappa Mu Epsilon*.

The officers for the year 1946–47 were: President, Terry D. McAdam; Vice-president, John W. Nipps; Secretary, Harold Snider; Treasurer, Miss Jean Badders; Corresponding Secretary, Miss Frances Breneman.

Those elected for the year 1947–48 are: President, Thomas Hotchkiss; Vice-president, Miss Jean Badders; Secretary, Miss Evelyn Hazlitt; Treasurer, Miss Jean Hobble; Corresponding Secretary, Miss Margaret Martinson; Sponsor, Miss Laura Z. Greene.

Mathematics Club, North Texas State Teachers College

Monthly meetings consisting of special programs, social activities and other functions were held by the *Mathematics Club* of North Texas State Teachers College during the year 1946–47.

The officers for 1947–48 are: President, Billy Workman; Vice-president, Kenneth Hannah; Secretary and Treasurer, Jack Standerfer; Reporter, Dan Spalding; Historian, Norma Harman; Sponsors, Dr. E. H. Hanson, Dr. J. B. Cooke, and Mr. Carl York.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

LECTURES BY DR. SYDNEY GOLDSTEIN

Dr. Sydney Goldstein, F.R.S., Professor of Mathematics at the University of Manchester, England, was presented at the University of Texas in a series of four lectures on the mathematics of transonic and supersonic airflow, on December 18, 19, and 20, 1947. Dr. Goldstein also delivered the Wright Brothers Lecture before the Institute of the Aeronautical Sciences in Washington, D. C., on December 17, 1947. Dr. Goldstein is Chairman of the Aeronautical Research Council of Great Britain.

OHIO CONFERENCE ON UNDERGRADUATE MATHEMATICS

Representatives of twenty liberal arts colleges met at Ohio Wesleyan University, Saturday, October 25, to hold a symposium on "Professional Problems in Teaching Undergraduate Mathematics in Liberal Arts Colleges." The con-

ference was organized by a small committee under the chairmanship of Professor Wayne Dancer of the University of Toledo. There were forty-three persons present, including representatives from Antioch College, Baldwin-Wallace College, Bowling Green State University, Capital University, University of Cincinnati, University of Dayton, Denison University, Heidelberg College, Hiram College, John Carroll University, Kent State University, Kenyon College, Marietta College, University of Michigan, Oberlin College, Ohio State University, Ohio Wesleyan University, University of Toledo, Western College, and Xavier University.

Professor S. A. Rowland presided during the first part of the conference when the group discussed the problem presented by the poor preparation in algebra of entering freshmen. Colleges which select their students carefully escape this problem, but schools which cannot apply selection, especially publicly-supported institutions, find that perhaps half the students enrolled in mathematics are not prepared to proceed with the traditional course in College Algebra. It seems to be common practice to give some sort of placement examination, requiring students making lower grades to take a course in intermediate algebra to correct their deficiency.

The next part of the discussion concerned the direction of work of superior students, Professor F. B. Wiley presiding. It was maintained that we do not properly challenge the ability of the better students. In some colleges these students are placed in separate sections for their freshman and sophomore mathematics, covering the same material as the other classes but doing a better job of it. Other ways suggested to develop the abilities of outstanding students were: an apprenticeship system in which a few students share the offices and certain tutorial duties with the faculty, enrollment in senior seminar courses, and encouragement to participate in undergraduate mathematics clubs. "Reading" courses seem to be out of vogue.

In the first part of the afternoon session Professor C. V. Newsom led the discussion on the ideal curriculum for a mathematics major. It was thought that the normal student specializing in mathematics should first take the customary sequence of courses through elementary calculus; then, in his junior and senior years study a year of advanced analysis (advanced calculus and differential equations), take a year's work in applications of mathematics (the exact courses to depend upon whether his interests lean towards physics, engineering, statistics, or teaching), some work in advanced algebra, a course in advanced geometry including the synthetic approach, and a course in the foundations or fundamental concepts of mathematics.

The group spent some time in discussing the problem: Should students who plan to take only one year of mathematics take the traditional courses in College Algebra, Trigonometry, and Analytic Geometry, or would they profit more from a one-year survey-type course covering a wider range of material? Although none present claimed to have found the complete answer to this question, there seems to be a trend toward the survey course for non-technical students.

Those with experience recommended that such a course include work on statistics and the mathematics of investment.

Professor Wayne Dancer took the chair for the discussion of opportunities for employment of young people who have specialized in mathematics for the bachelor's degree. It was recognized that the liberal arts college is not expected to prepare one for a vocation; nevertheless, students who want to major in mathematics also want to know how they can use their mathematics after graduation.

It was pointed out that mathematics is excellent training for subsequent professional study in many fields, and that an increasing number of professions are demanding a knowledge of advanced mathematics of their specialists. Several present reported that they had had calls from large industries and the Civil Service for college graduates who have majored in mathematics. Some corporations prefer to employ for many positions young people with a well-balanced liberal education, giving specific training themselves. In such cases a knowledge of mathematics is universally desirable.

To continue the study and discussion of problems raised, the group voted to ask the Ohio Section of the Mathematical Association of America to sponsor the work of this conference. An annual fall meeting for thi purpose met with general approval.

PERSONAL ITEMS

Fritz Herzog of Michigan State College has been promoted to an associate professorship.

Assistant Professor R. C. Hildner of the College of Wooster, Wooster, Ohio, has been appointed to an associate professorship at the University of New Mexico.

Dean C. C. Hurd of Allegheny College, Meadville, Pennsylvania, has accepted the position of theoretical research engineer at Carbide and Carbon Chemicals Corporation, Oak Ridge, Tennessee.

W. J. Jaffe of Newark College of Engineering has been promoted to an assistant professorship.

Assistant Professor Walter Jennings of Virginia Polytechnic Institute has been appointed to an assistant professorship at the Naval Postgraduate School, Annapolis, Maryland.

Associate Professor H. E. Jordan of the University of Kansas has been appointed to an associate professorship at Colby College.

Professor B. W. Jones of Cornell University will be on leave for the current academic year and will be at the California Institute of Technology.

A. R. Kirby of Fordham University has been promoted to an assistant professorship in the School of Medicine.

Dr. W. J. Kirkham of Oregon State College has been promoted to an associate professorship.

Professor H. D. Larsen of the University of New Mexico has been appointed to a professorship at Albion College, Albion, Michigan.

D. H. Lehmer, of the University of California has been promoted to a professorship.

Jerome C. R. Li of Oregon State College has been promoted to an assistant professorship.

Dr. W. S. Loud of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Minnesota.

Dr. A. K. Mitchell of Pratt and Whitney Aircraft Division of United Aircraft Corporation has been appointed to an associate professorship at the University of Maryland.

Dr. Josephine Mitchell has been appointed to an assistant professorship at Oklahoma A and M College.

Mr. A. G. Montgomery of the College of St. Thomas has been promoted to an assistant professorship.

Professor F. S. Nowlan of the University of British Columbia has been appointed to a visiting professorship at the Chicago Undergraduate Division of the University of Illinois for this year.

Assistant Professor R. E. Johnson of Mt. Holyoke College has been appointed to an associate professorship at Smith College.

Professor Rufus Oldenburger of the Illinois Institute of Technology is on leave of absence during the current year to complete a book on applied mathematics and to work as mathematician for the Woodward Governor Company, Rockford, Illinois.

Assistant Professor S. T. Parker of the University of Louisville has been appointed to an assistant professorship at Kansas State College of Agriculture and Applied Science.

Dr. P. K. Rees is now Associate Professor and Acting Head of the Department of Mathematics at Louisiana State University.

Associate Professor E. J. Purcell of the University of Arizona has been promoted to a professorship.

R. A. Rosenbaum of Reed College has been promoted to an associate professorship.

Assistant Professor C. E. Sealander of Iowa State College has been appointed to an assistant professorship at Ohio State University.

Abraham Seidenberg of the University of California has been promoted to an assistant professorship.

Assistant Professor W. S. Snyder of the University of Tennessee has been promoted to an associate professorship.

Professor C. F. Stephens of Prairie View University, Prairie View, Texas, has been appointed to a professorship at Morgan State College, Baltimore, Maryland.

W. M. Stone has been appointed to an assistant professorship at Oregon State College.

Otto Szasz of the University of Cincinnati has been promoted to a professorship.

H. P. Thielman of Iowa State College has been promoted to a professorship.

C. H. Vehse of West Virginia University has been promoted to a professorship.

Frantisek Wolf of the University of California has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Barnard College: Grace L. Bolton

Berea College: Ruth E. Porter

Haverford College: Dr. D. L. Thomsen

Los Angeles City College: R. E. Horton

Modesto Junior College: A. J. Osner

Oregon State College: J. F. Price, Mrs. Hazel Thurman, Mrs. Evelyn Schroeder, R. J. Wise

Queens College, CCNY: Dr. J. E. Eaton

Triple Cities College, Syracuse University: E. O. Allen

University of Alabama: Irma Berkowitz

University of Buffalo: Jean Lee Blaney, B. J. Clark, F. P. Kowalewski, Jr., Jane L. Noller, D. D. Strebe

University of California, Extension Division: Frank Harary

University of Michigan: P. S. Jones

University of Nebraska: Dr. Edwin Halfar, Dr. W. G. Leavitt

University of Oregon: Mrs. Pearl Van Atta

University of Tennessee: Ralph Donnell, Clyde Miller, Jack Moshman, Frances Street, Dr. R. L. Wilson

West Virginia University: Thomas Bauserman

G. W. Evans, a charter member of the Association, died in February, 1947.

Professor T. W. Wiley died September 1, 1947.

Mr. A. Z. Skelding wishes to dispose of back volumes of the MONTHLY covering a large part of the period from 1916 to date, some bound and some unbound. Anyone interested in purchasing such volumes should write for further information to Mr. A. Z. Skelding, Actuary, 162 Hamilton Road, Hempstead, L. I., N. Y.

The winner of the prize scholarship in the 1947 William Lowell Putnam examination is Mr. William Turanski of the University of Pennsylvania.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of South Carolina, Columbia, S. C., on Friday and Saturday, April 18–19, 1947. Professor Ruth W. Stokes, Chairman of the Section, presided at the Friday afternoon and Saturday morning meetings, and Professor W. L. Williams presided on Friday evening at the informal dinner given in honor of the visiting speaker, Professor L. M. Graves.

There were one hundred and forty-one present, including the following seventy-seven members of the Association: Louise Adams, E. A. Bailey, T. A. Bancroft, D. F. Barrow, Helen Barton, W. S. Beckwith, R. C. Blackwell, R. V. Blair, R. G. Blake, Floyd Bowling, M. G. Boyce, A. T. Brauer, T. C. Carson, B. G. Clark, F. E. Clark, E. G. Douglas, Nelle C. Douglas, Jeanette R. Durst, L. A. Dyke, F. A. Ficken, R. B. Folsom, Jack Frierson, L. L. Garner, W. H. Gaver, L. M. Graves, C. L. Hair, E. A. Hedberg, R. A. Hefner, Erik Hemmingsen, P. R. Hill, C. H. Holton, G. B. Huff, L. P. Hutchison, J. A. Hyden, L. W. LaGrone, G. B. Lang, J. W. Lasley, Jr., R. J. Levit, Anne L. Lewis, F. A. Lewis, R. A. Lytle, G. H. May, W. G. McGavock, S. W. McInnis, W. N. Mebane, Jr., H. A. Myer, D. D. Miller, W. B. Moye, H. M. Nahikian, Sara L. Nelson, J. D. Novak, Eugene Park, R. I. Pepper, I. E. Perlin, C. G. Phipps, Ellen F. Rasor, G. E. Reves, J. O. Reynolds, H. A. Robinson, L. B. Robinson, C. L. Seebeck, Jr., D. C. Sheldon, C. Eucebia Shuler, E. L. Stanley, L. W. Stark, A. L. Starrett, R. P. Stephens, Ruth W. Stokes, C. F. Strobel, Cora Strong, C. S. Sutton, Mary M. Templeton, J. M. Thomas, J. A. Ward, Betty R. Weber, W. L. Williams, Y. K. Wong.

At the business session the following officers were elected for the coming year: Chairman, J. W. Cell, North Carolina State College; Vice-Chairman, L. A. Dye, The Citadel; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its 1948 meeting at The Citadel.

The program consisted of the following papers:

1. *Generalized differential operators*, by Professor L. V. Robinson, University of South Carolina.

Taking for the argument the product of a function of x and the differential operator plus another function of x, several expressions involving functions of that argument operating upon other functions of x were derived. Likewise, corresponding expressions for inverse functions and functions of the inverse of the given argument were derived.

2. *On an equation all of whose roots are real*, by Professor Tomlinson Fort, University of Georgia, (by title).

The author discussed a little-known inequality that holds between the coefficients of an equation all of whose roots are real.

3. *The concyclic sets of points in the Morley configuration*, by Professor L. A. Dye, The Citadel.

The Morley configuration of a triangle consists of 27 points determined by 18 angle trisectors which lie six by six on nine lines. These lines are in three sets of parallels, making angles of 60° with one another. There are 144 circles, each of which passes through 4 or 5 of the 27+3 points of the figure. Professor Dye showed that there are 9 triples of Morley points concyclic with some pair of vertices of the triangle, and there are 54 other triples which are concyclic with one of the vertices. Further, there are 27 quadruples of Morley points which are concyclic with one of the vertices of the triangle and 54 other quadruples which are concylic.

4. *The non-existence of a certain type of odd perfect number*, by Dr. R. J. Levit, University of Georgia.

In this paper, Doctor Levit showed that no odd perfect number exists with the property that each of its prime power factors has a divisor sum which is itself a prime power, though all even perfect numbers do possess this property.

5. *Partial fractions with repeated linear factors in the denominator*, by Dean E. A. Bailey, LaGrange College.

Dean Bailey demonstrated certain properties of partial fractions, and the algebraic simplification of certain identities.

6. *A function with a finite discontinuity*, by Professor J. A. Ward, University of Georgia.

This paper appeared in the March, 1947, issue of this MONTHLY, page 162.

7. *Determination of a polynominal in several indeterminates by a finite set of its values*, by Mr. F. E. Clark, Duke University.

Mr. Clark proved that the coefficients of the complete polynomial f of degree p in n indeterminates are linear homogeneous polynomials in the values assumed by f at the set of points whose coördinates are the exponents of the various monomials of f. An application of this result to the proof of identities was given.

8. *A novel algorithm at the freshman level*, by Professor G. B. Huff, University of Georgia.

The algorithm presented was a device, discovered experimentally, for computing the values of a polynomial at a set of evenly spaced values of the arguments. The algorithm and its inverse apply to problems in theory of equations, interpolation and summation. Professor Huff stated that he had found the device useful as a challenge to enterprising undergraduates.

9. *A geometrical construction for a two star fix*, by Professor J. M. Thomas, Duke University.

Professor Thomas gave a ruler and compass construction in the plane for finding latitude and longitude from the declinations, Greenwich hour angles, and altitudes of two points on the celestial sphere.

10. *Undergraduate mathematics curricula*, by Professor L. M. Graves, University of Chicago.

In this paper, Professor Graves discussed some principles believed to be advantageous in the setting up of a curriculum in mathematics. Among these were: emphasis on the vector point of view in trigonometry and analytic geometry, as well as in later courses; emphasis on relations between different concepts and methods; and training in logic and postulation methods. Some changes and innovations already operating at the University of Chicago were mentioned.

11. *Exact probabilities for the common tests of statistical hypotheses*, by Professor T. A. Bancroft, University of Georgia.

Certain probability distribution functions were transformed into the incomplete beta-function ratio. A general formula and certain special cases derived by Professor Bancroft were given for obtaining values of the incomplete beta-function ratio outside the range of the existing tables.

12. *The jeep problem: a more general solution*, by Professor C. G. Phipps, University of Florida.

This paper appeared in the October, 1947, issue of this MONTHLY.

13. *An exact formula for the quoted price of a bond*, by Dr. P. M. Hummel and Dr. C. L. Seebeck, University of Alabama.

By the use of present values of future benefits, a formula was established for the quoted price of a bond to yield a given rate.

14. *A solution to the bracelet problem*, by R. G. Blake, University of Florida.

A solution of the bracelet problem was given by stringing together x_1 beads of one kind, x_2 of a second kind, and so forth. The problem was made to depend upon the number of cyclic permutations and upon the consideration of which permutations were symmetric.

15. *Functional analysis*, by Professor L. M. Graves.

After giving illustrative examples of functionals and functional equations, Professor Graves reviewed the differential and integral calculus in Banach spaces, and the fundamental theorems on implicit functions. A simple device was used to procure a theorem on the extent of the domain of functions defined implicitly. This theorem may be applied to show the existence of solutions, satisfying prescribed boundary conditions, of certain classes of nonlinear ordinary or partial differential equations.

16. *On Transon's formula for aberrancy*, by Professor J. W. Lasley, Jr., University of North Carolina.

Abel Transon in 1841 developed the concept of aberrancy, or deviation, called by him the second "affection" of curvature. In his analytical argument, Transon used the inclination of the tangent for the independent variable rather than the arc length of the curve. If the latter is used, Professor Lasley demonstrated it was possible by means of a one parameter family of equilateral hyperbolas to obtain Transon's formula by direct calculus methods, and in a simpler form.

17. *On the development of the idea of dimension*, by Dr. Erik Hemmingsen, University of Georgia.

The development of the concept of dimension was traced from the ideas implied by Euclid to the definitions used by the topologist. Each definition discussed was shown to have certain disadvantages.

18. *A derivation of the equations for vibrating string*, by Professor F. A. Ficken, University of Tennessee.

Under rather general assumptions, the derivation was examined in order to display what use was made of each hypothesis. The longitudinal component of the displacement satisfied an equation of Sturm-Liouville type in which each of the variable coefficients was present and had immediate physical significance.

19. *On the characteristic equation of certain matrices*, by Professor A. T. Brauer, University of North Carolina.

Professor Brauer gave an algebraic proof of a theorem on matrices obtained by von Mises from results in the theory of probability. The paper was published in the *Bulletin of the American Mathematical Society*, June, 1947.

20. *Derivation of the impedance circle diagram as a mapping problem*, by W. R. Mullin, Vanderbilt University, introduced by Professor J. A. Hyden.

The derivation of the impedance circle diagram for the computation of input impedance of a lossless transmission line was accomplished in this paper as a mapping problem rather than from electrical considerations. Certain facts concerning transmission lines were shown to follow directly from the mathematical derivation with the introduction of a minimum amount of electrical theory.

21. *Determinants whose columns (rows) are given sequences*, by Professor I. E. Perlin, Georgia School of Technology.

Professor Perlin proved certain theorems and gave certain interesting examples relative to determinants whose columns were composed of given sequences.

22. *A topic of infinite matrices*, by Professor Y. K. Wong, University of North Carolina.

This paper dealt with the reducibility of a linear transformation in Hilbert space in relation to its inverse. Examples were constructed to show that infinite matrices may have reciprocals without the property of reducibility.

23. *The plane projectivity between points connected by an associativity*, by Professor B. G. Clark, Vanderbilt University.

Using the fact that the vertices of two triangles in the plane are associated with their opposite sides and that two sets of points associated with the same set of lines are projective to each other, Professor Clark discussed the various types of non-singular collineations determined by six points in their several relative positions.

24. *A plane transformation determined by opposite edges of a tetrahedron*, by Professor J. M. Clarkson, North Carolina State College, introduced by the Secretary.

Professor Clarkson illustrated a method for a plane transformation by considering a tetrahedron whose three sets of opposite edges were parallel to the coördinate planes. The equations of transformation he used showed that the straight line loci of the xy plane were mapped into hyperbolas on the xz plane. Some of these hyperbolas were composite, and the points of the xy plane which caused the breakdown were singular points in the transformation.

25. *Maxima and minima of sums*, by Professor Tomlinson Fort, University of Georgia, (by title).

The calculus of variations is concerned with the problem of maxima and minima of definite integrals. The definite integral is the limit of a sum. The present paper treats maxima and minima of sums by methods not dissimilar to those commonly applied to the integral.

H. A. ROBINSON, *Secretary*

MAY MEETING OF THE NEBRASKA SECTION

The twenty-third meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska, in Lincoln, on May 3, 1947. Professor Ralph Hull, Chairman of the Section, presided.

The attendance was twenty-four, including the following thirteen members of the Association: M. A. Basoco, H. W. Becker, E. M. Berry, A. K. Bettinger, W. C. Brenke, C. C. Camp, H. M. Cox, J. M. Earl, M. G. Gaba, C. B. Gass, E. H. Hadlock, Ralph Hull and Lulu L. Runge.

At the business meeting the following officers were elected for the coming year: Chairman, H. M. Cox, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska.

The following papers were presented:

1. *An application of the Laplace transformation to the solution of a certain boundary value problem*, by D. D. Rippe, University of Nebraska, introduced by the Chairman.

The speaker considered the differential equation

$$\theta_{xx}(x, t) = \frac{1}{K}\theta_t(x, t) \qquad t > 0, -a < x < a,$$

with boundary values

$$\theta(x, 0) = 0 \qquad -a < x < a, \qquad \theta(a, t) = \theta(-a, t) \qquad > 0$$

$$Q(a, t) = 2\int_0^a \theta(x, t)dx \qquad t > 0,$$

$$Q_t(a, t) = c[\theta_m - \theta(a, t)] \qquad t > 0.$$

He showed, by applications of the Laplace transformation and especially by the application of the theory of the complex inversion integral which arises therein, that it was possible to express the solution in the form

$$\theta(x, t) = \theta_m\left[1 - \sum \frac{e^{-kr_n^2/l}\cos(r_n x/a)}{r_n(1 + \delta + \delta^2 r_n^2)\sin r_n}\right]$$

where r_n is a root, z_n, of the transendental equation $\cot z = 2kz/ca$ and $\delta = 2k/ca$.

2. *On an integral equation with discontinuous kernel*, by C. C. Camp, University of Nebraska.

Reference was made to the paper by J. D. Tamarkin in Vol. 29 of the *Transactions of the American Mathematical Society*, in which he considered the equation $u(x) = \rho\int_0^1 k(x, \xi)u(\xi)d\xi$. In this the kernel has a jump of one across the line $x = \xi$. Recently the author has treated the case of such a jump across the secondary diagonal $x = 1 - \xi$ in addition, but in a partial derivative of $k(x, \xi)$. Professor Camp extended the theory to discontinuities in the kernel itself along both diagonals. He showed that, under certain restrictions, the associated expansion theory still holds.

3. *Problems in the algebra and topology of electrical circuits*, by H. W. Becker, Omaha, Nebraska.

The speaker showed the importance of the operation $a \| b = ab/(a+b)$ in electrical cultures. It generates one of an infinite number of functional arithmetics defined by $F(a, b) = a \oplus b$.

The easiest method yet devised for the solution of bridge circuits was established. The transfer

conductance G of any branch b is the fracto-sum of the G's of the two component circuits formed by opening and shorting any other branch b. That is,

$$G_b = G_{b0} \otimes G_{b00} = N_b/D_b = \frac{b'N_{b00} + N_{b0}}{b'D_{b00} + D_{b0}}.$$

The enumeration of bridge circuits is made difficult by the possibility of topological identity between apparently different configurations, and also of symmetries. Topologically generalized binomial coefficients are required, in which the combinations are reduced by any symmetries present. Symmetries also create topological products $C_m \otimes C_n \neq C_m \cdot C_n$ when $m = n$. The census of all bridge circuits of less than thirteen branches, basic or unrestricted, was tabulated under various classifications.

4. *A new approach to celestial navigation*, by O. C. Collins, University of Nebraska, introduced by the Secretary.

In this paper, it was remarked that for any two stars there exist on the celestial sphere two families of ellipse-like loci such that for the one family the sums of the star distances on the sphere are constants and for the other family the differences are constants. For special values of the parameters, these loci degenerate into a system of three mutually perpendicular great circles, one of which passes through the two star positions and another of which is equidistant from the two star positions. A point on the sphere at measured distances from the two star positions may be located with respect to the three-great-circle system of those two stars in terms of the lengths of two mutually perpendicular great circle arcs, whence its equatorial coördinates may be found if the positions of the stars are known.

5. *Geometry in college*, by C. A. Huck, Peru State Teachers College, introduced by the Chairman.

The speaker called attention to the increased attendance in mathematics classes following World War II. He remarked that, following World War I, mathematical study was placed in an unfavorable position, and urged that we now organize to avoid a recurrence of what happened at that time. He urged that mathematics be presented as a dynamic force in the development of all phases of life. Because of inadequate preparation, high school teachers are not capable of inspiring their students to continue the study of mathematics beyond the second year. In order to better prepare these teachers, colleges should offer a course in plane geometry, as well as college algebra.

6. *Note on the effect of postponement of college education by war emergency*, by H. M. Cox, University of Nebraska.

In this address Professor Cox reviewed the results compiled from the examinations routinely administered to all entering sophomore and freshman students, covering subjects such as mathematics, science, English usage, social studies, science information, and college aptitude. By analysis of variance, no significant differences on any one of the tests were found among either men or women students when they were grouped by year of graduation, namely 1940 through 1945.

There were, however, significant differences between the separate groups just described and the graduates of high school in 1946. Extreme variations occurred in all groups studied and on all of the examinations. The veterans group averaged lower than the normal group of entering freshmen, but an exploratory study of the achievement of veterans indicated equal chances of success for the older and younger veterans within the age range of 18 to 23.

7. *Sequential analysis in statistics*, by Ralph Hull, University of Nebraska.

Professor Hull gave an expository talk on recent developments in this field, due to Wald and others.

LULU L. RUNGE, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The fifteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin, Madison, on Saturday, May 3, 1947. Professor R. H. Bruck presided at the morning session, Professor R. H. Bardell at the afternoon session, and the Section Chairman, Professor H. P. Evans, presided at the business meeting.

There were forty in attendance, including the following twenty-four members of the Association: L. K. Adkins, K. J. Arnold, R. H. Bardell, Leon Battig, May M. Beenken, W. W. Begelow, R. H. Bruck, B. H. Colvin, H. P. Evans, E. G. Harrell, J. F. Kenney, S. C. Kleene, R. E. Langer, H. W. March, Morris Marden, Sister Mary Felice, F. E. Nemmers, Elli Otteson, G. A. Parkinson, H. P. Pettit, G. F. Rose, Abraham Spitzbart, J. I. Vass, and R. L. Wilson.

At the business meeting, the following officers were elected for the coming year: Chairman, R. C. Huffer, Beloit College; Secretary, P. L. Trump, University of Wisconsin; Program Committee, H. P. Pettit, Marquette University, Etylwynn Beckwith, Milwaukee-Downer College, E. G. Harrell, Platteville State Teachers College. The executive committee was instructed (1) to keep itself and the Section informed concerning proposed general changes in standards of mathematical curriculum or of teaching personnel; (2) to coöperate with other groups in promoting high educational standards in Wisconsin; (3) to use every suitable occasion for securing favorable publicity for mathematics in Wisconsin. It was voted to accept the invitation of Professor H. H. Conwell of Beloit College to hold the next meeting on May 8, 1948 at Beloit.

The following papers were presented:

1. *Industrial applications of high energy electron accelerators*, by Jack Wilson, Allis Chalmers Corporation, introduced by Dr. Louis A. Wolf.

2. *The word problem*, by Professor S. C. Kleene, University of Wisconsin.

The problem as proposed was based upon the assumption that a finite set of symbols, such as a, b, l, is given, with words constructed from them, such as $abbla$. Let a finite number of equations, such as $bbl=ab$, be postulated between parts of words, so that then $abbla=aaba$, $bblb=ablb$, and so on. Axel Thue in 1914 proposed the problem, to find a method by which one can always decide whether two given words are equal. To be certain that one can always decide means that the decision must depend solely on the outcome of a computation. Using the theory of computing machines given by A. M. Turing in 1936, Emil Post in the March 1947 *Journal of Symbolic Logic* proved that the above problem of Thue is unsolvable. Professor Kleene explained Turing's and Post's methods. The problem proved unsolvable is the same as the word problem for semi-groups, and the possibility appears herewith that the word problem for groups may likewise be unsolvable.

3. *Shock hydrodynamics*, by Professor J. O. Hirschfelder, University of Wisconsin, introduced by Professor H. P. Evans.

The equations of hydrodynamics for non-viscous fluids were derived from Newton's laws. It was shown that, whereas the Lagrangian form of the equations appears to be linear, the Eulerian forms are non-linear. In all practical problems it is the Eulerian form which is used. These equations permit two types of solutions, namely continuous and discontinuous. The possibility of the discontinuities of the solutions is due to the non-linearity of the Eulerian equations. Riemann developed a method of characteristics for solving one-dimensional hydrodynamical problems when-

ever the solution is continuous. The continuous solutions are characterized by a constancy of the entropy for the individual elements of the fluid. If the fluid is compressed, the Riemann characteristics come together and the solution becomes discontinuous. The planes of the discontinuities are known as shocks. There is a finite and calculable entrophy change of the fluid element in passing across the shock. A continuous pressure pulse will develop a discontinuity after a certain length of time. The mathematical behavior of the interaction of shock waves is still open for investigation, and many strange effects are known experimentally to occur which may be traced directly to the non-linearity of the hydrodynamical equations.

PAUL L. TRUMP, *Secretary*

MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The May Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the George Washington University, Washington, D. C., on Saturday May 3, 1947. Professor W. K. Morrill, Chairman of the Section, presided.

The attendance was sixty including the following forty-two members: T. E. Berry, W. E. Bleick, H. H. Campaigne, C. R. Clark, Abraham Cohen, J. F. Daley, Alexander Dillingham, J. A. Duerksen, E. J. Finan, N. J. Fine, Ansel Fisher, M. K. Fort, Jr., B. C. Getchell, Michael Goldberg, R. A. Good, D. W. Hall, G. A. Hedlund, M. A. Hyman, Ernest Johnston, F. E. Johnston, Sidney Kaplan, L. M. Kells, V. L. Klee, Jr., W. D. Lambert, A. E. Landry, Carol V. McCamman, Florence M. Mears, A. B. Mewborn, W. K. Morrill, W. H. Norris, Jr., E. K. Paxton, C. E. Rhodes, J. N. Rice, E. D. Schell, W. F. Shenton, A. D. Sollins, J. H. Taylor, J. Tyler, A. W. Tucker, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

At a business meeting held during the afternoon session the following officers were elected for the coming year. Chairman, E. J. Finan, Catholic University; Secretary-Treasurer, M. H. Martin, University of Maryland; additional members of the Executive Committee, G. A. Bingley, St. John's College, and T. W. Hatcher, Virginia Polytechnic Institute. It was announced that D. W. Hall, University of Maryland, had been elected Sectional Governor.

The first four of the following papers were read at the morning session. Professor Tucker's paper was read at the afternoon session at the invitation of the Section.

1. *A geometrical approach to Frandtl-Meyer flow*, by Professor M. H. Martin, University of Maryland.

In two-dimensional, steady, irrotational, isentropic flows for which the velocity potential surface $\phi=\phi(x, y)$ is a developable surface, the Legendre transformation which carries the partial differential equation

$$(G^2-u^2)\phi_{xx}-2uv\phi_{xy}+(G^2-v^2)\phi_{yy}=0$$

into a linear partial differential equation (L) breaks down. In the investigation of this exceptional type of flow if the developable surface is viewed as the envelope of a parameter family of planes $\phi=ux+vy-\omega$, where u, v, ω are functions of a parameter τ, the following facts are easily established: (i) The curves $u=u(\tau)$, $v=v(\tau)$ in the hodograph plane are epicycloids, the characteristics of (L); (ii) The flow in the physical plane is of Pradtl-Meyer type, the Mach lines being the projections of the generators of the developable surface upon the physical plane; (iii) The angle between

the direction of flow and a Mach line is the Mach angle; (iv) The equipotential lines are presented parametrically by

$$Z = Me^{i\theta} + \frac{\phi}{l} e^{i(\theta-\alpha)}.$$

in which $M = W + iW$; where W is an arbitrary function of the parameter θ (the direction of flow) and l is a determined function of θ having geometrical significance.

2. *Some properties of pointwise periodic transformations*, by E. E. Floyd, University of Virginia, introduced by the Secretary.

Let X be a compact metric space and $f(X) = X$ a pointwise periodic transformation on X. We say that f on X gives a continuous orbit decomposition at $x \in X$ if $x_i \to x$ implies $\lim O(x_i) = O(x)$, where $O(y)$ denotes the orbit of y. It is shown that, if $x \in X$ is a non-isolated point, then there exists a sequence $x_i \to x$ such that $\lim O(x_i) = O(x)$. Using this as a lemma, it is shown that if f does not give a continuous orbit decomposition at $x \in X$, then there exists an invariant perfect set K, $K \subset X$ and $x \in K$, such that $x_i \to x$, $x_i \in K$, implies $\lim O(x_i) = O(x)$.

3. *Leonard and Thomas Digges—their contribution to practical measurement*, by Professor W. F. Shenton, American University.

This study of the lives and mathematical contributions of this father and son team of Elizabethan mathematicians springs from a consideration of the interesting woodcuts in the copy of the 1591 *Pantometria* from the Artemas Martin Library. It shows, by means of lantern slides, the construction and use of the quadrant, square, and theodolite in working simple surveying problems. As to their place in mathematical performance, Mr. Halliwell says "Thomas Digges ranks among the first English mathematicians of the sixteenth century. Although he made no great addition of science, yet his writings tended more to its cultivation than perhaps all those of other writers on the same subjects put together."

4. *Formulas for numerical quadrature*, by Professor E. J. McShane, University of Virginia.

By integration of well-known interpolation formulas, numerical quadrature formulas are developed for computing $y(x)$ from the tables of $y'(x)$, or of $y''(x)$, or of both. One of these is useful as a check formula when using Simpson's rule. Another (due to C. B. Morrey) is useful when both y' and y'' are tabulated; it is easy to use and is more accurate than Simpson's rule. Another gives the second difference of y in terms of y''. These formulas are combined to furnish a procedure for the numerical integration of second-order differential equations, or of systems of such equations.

5. *The use of transformations in college geometry*, by Professor A. W. Tucker, Princeton University.

Elementary transformations and their products can play the unifying role in college geometry that elementary functions and their compounds play in college algebra and calculus. Reflections, inversions, pole-polar reciprocations, and their products (operating on the material of ordinary plane geometry) can be used to construct various geometries that belong naturally to the *Erlanger Programm* of Felix Klein. Reflections generate the displacements and symmetries of the euclidean geometry of congruence; reflections and inversions generate the direct and inverse circular transformations of conformal geometry; reflections and pole-polar reciprocations (in circles) generate both the collineations and correlations of projective geometry. Of course, the geometric forms and quantities (such as cross-ratio) that are left invariant by a whole group of transformations are precisely those preserved by the elementary transformations used to generate the group. The need for a transformation to be one-to-one without exception leads to the invention of a single point at infinity in conformal geometry and of a line of points at infinity in projective geometry. Certain

"subgeometries" of conformal geometry and projective geometry arise very simply from restrictions on the transformations that leave the point at infinity fixed, hyperbolic non-euclidean geometry from the circular transformations that preserve the interior and circumference of one particular "fundamental" circle, affine geometry from the collineations that preserve the line of points at infinity, and elliptic non-euclidean geometry from collineations related by central projection to spatial transformations generated by reflections in planes through the particular center of projection.

E. J. FINAN, *Secretary*

MAY MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, May 10, 1947. Dr. Guy Stevenson, Chairman of the Section, presided.

There were fifty-four in attendance, including the following twenty-one members of the Association: M. C. Brown, P. P. Boyd, H. H. Downing, Clarence Ford, A. A. Grau, Charles Hatfield, Sr., Aughtum S. Howard, W. R. Hutcherson, S. J. Jasper, E. D. Jenkins, C. G. Latimer, D. J. Myatt, R. S. Park, Sallie E. Pence, D. W. Pugsley, G. G. Roberts, W. J. Robinson, F. V. Rohde, J. H. Simester, D. E. South, Guy Stevenson.

At the business meeting the following officers were elected for the next year: Chairman, D. W. Pugsley, Berea College; Secretary, Sallie E. Pence, University of Kentucky. The annual meeting in 1948 is to be held at Berea College.

The following papers were presented:

1. *Trisection of an angle by means of higher plane curves*, by P. P. Boyd, University of Kentucky.

Dean Boyd gave two methods for the graphical trisection of an angle by means of higher plane curves. He first reduced the problem of the intersection of the type cubic with a conic to the intersection of two conics, and next presented the solution of de Longchamp's (1888). He concluded by mentioning the nature of the problem and the proofs of Klein and Dickson.

2. *Development of the Frenet formulas for N-dimensions*, by S. J. Jasper, University of Kentucky.

The well known three dimensional vector forms of the Frenet formulas were extended to a flat space of N-dimensions. The speaker developed the method of finding the $(N-1)$ unit normal vectors and the $(N-1)$ curvatures associated with these vectors for a curve in N-dimensions.

3. *Contrasting two solutions of a certain problem in modern geometry*, by W. R. Hutcherson, Berea College.

Given any triangle ABC, through the three vertices lines AL, BM, and CN were drawn through any point O to points L, M, N on the opposite sides. Two proofs were given for the formula

$$\frac{OL}{AL}+\frac{OM}{BM}+\frac{ON}{CN}=1.$$

The first proof was a lengthy one depending upon Ceva's theorem. The second one was quickly accomplished by consideration of the areas of certain triangles.

4. *Applications of mathematics in meteorology*, by McClellan Cook, Jr., University of Louisville, introduced by Dr. Stevenson.

The speaker discussed practical applications of formulas for forecasting the movement of isobars, isallobars, pressure centers, and fronts, and for forecasting the acceleration of pressure systems and the intensifications of pressure systems.

5. *The graphic construction of a lunar eclipse*, by F. V. Rohde, University of Kentucky.

After defining the astronomical terms involved, the computations necessary for the graphic construction of an eclipse from elements, as given in the American Ephemeris and Nautical Almanac, were explained and an actual construction made for the eclipse of June 3, 1947. Eclipse limits and the conditions that must be satisfied before a lunar eclipse can occur were discussed briefly.

M. C. BROWN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, May, 8 1948
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS, Atchison, April 10, 1948
KENTUCKY, Berea, Kentucky, May, 1948
LOUISIANA-MISSISSIPPI, Lafayette, La., April 23–24, 1948
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, April 24, 1948
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 1, 1948
NORTHERN CALIFORNIA
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Philadelphia, Pa., Nov. 27, 1948
ROCKY-MOUNTAIN
SOUTHEASTERN, Charleston, S. C., March 19–20, 1948
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
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ALGEBRA FOR COLLEGE STUDENTS by Britton and Snively (529 pages, $3.00), which was published in April, 1947, has been the subject of much comment by teachers. They mention its "lucidity and precision," "its ample problems," "clearly-stated definitions," and note that "the review chapters are particularly good for the veterans. . . ." The first half of the text consists of twelve chapters which cover the fundamental ideas of algebra. The last half of the book develops the standard topics of the college algebra course with stress on important underlying ideas. *INTERMEDIATE ALGEBRA* (337 pages, $2.00), by Britton and Snively, consists of the first twelve chapters of *Algebra for College Students,* with chapters on Logarithms, Progressions and the Binomial Theorem, and Systems Involving Quadratic Equations.

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